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RELATED SOLUTIONS TO THE PERSPECTIVE THREE-POINT POSE PROBLEM

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Abstract: The Perspective Three-Point Pose Problem (P3P) involves solving Grunert's system of quadratic equations for the distances from the center of perspective to the three control points, typically resulting in multiple mathematical solutions. Relationships between the corresponding possible camera positions in space have only rarely been studied. Several efforts have been made though to understand the number of solution points using various assumptions. In this article, the number of solutions is determined in the limiting case where the center of perspective is far from the plane containing the control points, as compared with its distance to the danger cylinder. Moreover, concise formulas are given for the other solutions based on a knowledge of one of the solutions. It turns out that the projection onto the control points plane of the various solution points lie at the intersection points of two rectangular hyperbolas. A certain deltoid curve also plays a crucial role.

1 INTRODUCTION

When a camera image includes three points that correspond to three points in space with known positions, the position of the camera can nearly be deduced from this scant information. The mathematical details for doing so have long been understood, but unfortunately, solving the relevant system of equations can result in up to four solutions, only one of which is the actual position of the camera.

In this setup, the three known points are referred to as "control points," the camera's optical center (assuming a pinhole camera model here) is the "center of perspective," and the system of equations is "Grunert's system." Assuming, as we will, that the three control points are not collinear, they will instead lie an a particular circle. By extending this circle in the direction perpendicular to the plane containing the control points, one obtains the "danger cylinder," which has importance in the analysis of the problem, known as the "Perspective 3-point pose problem (P3P)".

The problem was originally stated and solved in (Grunert, 1841). Since then, various other algebraic methods have been discovered for solving it, generally requiring some sort of fourth degree polynomial. The state of the art as of the twentieth century is very well documented in (Haralick et al., 1994).

The issue of repeated solutions to Grunert's system and the weaker circumstance of repeated roots to Grunert's quartic polynomial has been explored by a number of researchers in recent years. There has been a good deal of interest too in determining the number of positive real solutions to the system, based on specific values of its parameters. (Wolfe et al., 1991) provides some excellent geometric insight into these matters by examining various configurations of triangles. A sufficient condition for four positive solutions is given in (Zhang and Hu, 2005). In (Zhang and Hu, 2006) the same authors explore the danger cylinder and use a certain Jacobian determinant to establish that this is where repeated solutions to Grunert's system occur.

(Gao et al., 2003) solves the difficult problem of classifying the number of real solutions and the number of positive solutions, depending on the values of the parameters. Unfortunately this work does not provide much geometric insight into this issue. (Tang et al., 2008) gives a better geometric sense of some of the conditions. (Yang, 1998) and (Faugère et al., 2008) provide algorithms that can assist in the same classification problem.

(Rieck, 2014) is an attempt to move away from the direct study of Grunert's system, involving various distances, and focuses more on the position of the center of perspective in relation to the control points. This has the potential of eventually answering various P3P questions in a satisfying geometric manner. As a tangible step towards this goal, the present article achieves a complete understanding of the various mathematical solutions in the situation where the center of perspective is sufficiently far from the plane containing the control points, as compared to its distance from the danger cylinder. The analysis of this situation involves the appearance of a classical curve known as a "deltoid."

2 PROBLEM STATEMENT

2.1 mathematical formulation

The assumptions and notation of (Rieck, 2014) will be used throughout. The basic setup can be seen in Figure 1, and is described as follows. It is assumed that a coordinate system is chosen for which the three control points, P_1, P_2, P_3, lie on the unit circle centered about the origin in the xy-plane. The danger cylinder is thus given by the equation

$$x^2+y^2=1.$$

For $j=1,2,3$, let

$$(x_j,y_j,0) \;=\; (\cos\phi_j,\sin\phi_j,0)$$

be the coordinates of P_j, with $-\pi<\phi_j\le\pi$. Let

$$t_j \;=\; \tan(\phi_j/2) \;=\; y_j/(1+x_j),$$

so that

$$x_j \;=\; (1-t_j^2)/(1+t_j^2)$$

and

$$y_j \;=\; 2t_j/(1+t_j^2).$$

For $j=1,2,3$, let d_j be the distance between the two control points other than P_j. The center of perspective will be denoted p, and r_j will be the distance from p to P_j ($j=1,2,3$). The (unknown) coordinates of p will just be denoted (x,y,z). For $j=1,2,3$, let

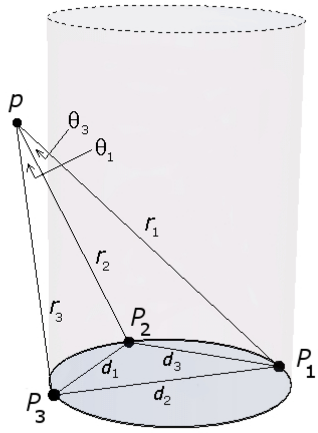

Figure 1: the setup

θ_j be the angle at p created by the two rays to the two control points other than P_j. These angles are presumed to be known since they are easily computed from the camera images of the control points and the camera intrinsics. Let $c_j=\cos\theta_j$.

Finding the coordinates (x,y,z) of the center of perspective p now becomes a matter of solving a pair of systems of equations, as follows:

$$\left\{\begin{array}{rcl} r_2^2+r_3^2-2c_1r_2r_3 &=& d_1^2\\ r_3^2+r_1^2-2c_2r_3r_1 &=& d_2^2\\ r_1^2+r_2^2-2c_3r_1r_2 &=& d_3^2 \end{array}\right. \tag{1}$$

$$\left\{\begin{array}{rcl} (x-x_1)^2+(y-y_1)^2+z^2 &=& r_1^2\\ (x-x_2)^2+(y-y_2)^2+z^2 &=& r_2^2\\ (x-x_3)^2+(y-y_3)^2+z^2 &=& r_3^2 \end{array}\right. \tag{2}$$

Since finding the unknowns r_1, r_2, and r_3 serves only an intermediary role in the actual goal of determining (x,y,z), one might consider using classical methods to eliminate them. Without care though, this can produce some very complicated equations. However, in (Rieck, 2014), a few reasonable and interesting equations were discovered that relate x, y and z directly to the known parameters d_1, d_2, d_3, c_1, c_2, and c_3.

We now turn our attention to understanding the other solution points if we assume that a particular solution point is known. Of course, these other solution points could be obtained from the combined system (1) and (2). Our goal though is not to just obtain these other solution points, but to understand their algebraic and geometric relationships with the given solution point. This issue will be our focus henceforth.

P will denote some "reference point," with coordinates (X,Y,Z), presumed known. When (X,Y,Z)

is substituted for (x,y,z) in (2), and solved for r_1, r_2 and r_3, and when these values are then used in (1), the quantities c_1, c_2 and c_3 can be readily determined. We now regard that P is the particular known solution point to (1) and (2) for these values of c_1, c_2 and c_3.

Now, these values of c_1, c_2 and c_3 are associated with P, and these will be understood to be fixed, as long as the point $P = (X,Y,Z)$ is fixed. The combined system (1) and (2) is now investigated for these values of c_1, c_2, and c_3. Another solution point p with coordinates (x,y,z) will be said to be "related to" the reference point P. Though the parameters c_1, c_2, and c_3 are the same for P and p, the corresponding r_1, r_2, and r_3 values are, in general, different. Ordinarily, P is not considered to be related to itself. However, when P is a repeated solution to the system of equations, it is prudent to regard that P is related to P.

For an arbitrary reference point, the problem of understanding, as fundamentally as possible, the relation between it and its related points seems unwieldy, at present. So instead we will consider this problem only when the reference point P is sufficiently far from the plane containing the control points, as compared with its distance to the danger cylinder. More precisely, we will suppose that the quantity $|1-X^2-Y^2|/Z^2$ is sufficiently small. We will obtain precise and interesting formulas for (x,y) in the limit as $|1-X^2-Y^2|/Z^2 \to 0$. Though this is a limiting setup, it can nevertheless shed considerable light on setups where $|1-X^2-Y^2|/Z^2$ is merely "reasonably" small.

2.2 a simple example

A particularly simple example of the sort of problem being considered in this article will now be examined in detail. The reference point coordinate Z will be kept general, and we will later consider what happens as Z is allowed to grow without bound. We will however fix $X = 0$ and $Y = 0$.

To further simplify this example, the control points will be assumed to form the vertices of an equilateral triangle. Let us specifically take $(x_1,y_1) = (1,0)$, $(x_2,y_2) = (-1/2,\sqrt{3}/2)$ and $(x_3,y_3) = (-1/2,-\sqrt{3}/2)$. This means that the distance between any pair of control points is $\sqrt{3}$.

Clearly $c_1 = c_2 = c_3$, and we will let c denote this common value. Let us now consider related solution points (x,y,z). System (1) becomes just

$$\begin{cases} r_2^2+r_3^2-2\,c\,r_2r_3 \;=\; 3 \\ r_3^2+r_1^2-2\,c\,r_3r_1 \;=\; 3 \\ r_1^2+r_2^2-2\,c\,r_1r_2 \;=\; 3. \end{cases}$$

By subtracting pairs of equations, we obtain

$$\begin{cases} (r_2+r_3-2cr_1)(r_2-r_3) \;=\; 0 \\ (r_3+r_1-2cr_2)(r_3-r_1) \;=\; 0 \\ (r_1+r_2-2cr_3)(r_1-r_2) \;=\; 0. \end{cases}$$

Of course, one solution has $r_1 = r_2 = r_3$. Denoting this common value r, we obtain

$$c = \frac{2r^2-3}{2r^2}.$$

This solution though simply corresponds to the reference point $(0,0,Z)$, with $r^2 = 1+Z^2$.

But there are three other solutions, corresponding to three related points. For instance, by setting $r_2 = r_3 = r$ and $r_1 = (2c-1)r$ (same r as before), we obtain another solution. The coordinates (x,y,z) of the corresponding point can then readily be found by solving system (2), which simplifies to

$$\begin{cases} (x-1)^2+y^2+z^2 \;=\; (2c-1)^2\,r^2 \\ (x+\frac{1}{2})^2+(y-\frac{1}{2}\sqrt{3})^2+z^2 \;=\; r^2 \\ (x+\frac{1}{2})^2+(y+\frac{1}{2}\sqrt{3})^2+z^2 \;=\; r^2. \end{cases}$$

From here, it is straightforward to deduce that

$$\begin{cases} x \;=\; (2r^2-3)\,/\,r^2 \;=\; (2Z^2-1)\,/\,(Z^2+1) \\ y \;=\; 0 \\ z^2 \;=\; (r^2-3)^2\,(r^2-1)\,/\,r^4 \\ \quad\;=\; Z^2\,(Z^2-2)^2\,/\,(Z^2+1)^2. \end{cases}$$

Now, finally, notice that as $Z \to \infty$, the projection of the point (x,y,z) onto the xy-plane is (2, 0). The other two solution points can be similarly obtained.

3 PROBLEM SOLUTION

3.1 preliminaries

Henceforth $P = (X,Y,Z)$ will be the reference point, and it is presumed to be known. A related solution point, for the same parameters (d_1, d_2, d_3, c_1, c_2, and c_3), will generically be denoted $p = (x,y,z)$. However, since it is of no consequence, we will always assume that $Z \geq 0$ and $z \geq 0$. We will be especially interested in the case where Z is large.

There is a temptation here is to suppose that since the camera is now so far from the control points, we can now switch from a pinhole camera model to a model that instead uses orthogonal projection. This is wrong for two reasons. Firstly, P3P is inherently concerned with the angles between lines through the camera's center of perspective. Secondly, understanding the mathematics involved as Z approaches infinity is only a useful way of coming to an appreciation of the approximate behavior of P3P when $|1-X^2-Y^2|/Z^2$ is "reasonable small." There is certainly a vagueness in this phrase, but it can be asserted that the general behavior in the limiting case is manifest as well at relatively short ranges.

The P3P solutions in the limiting case turn out to be quite surprising and interesting. This can already be somewhat anticipated from the "simple example." We will require the following basic fact.

Lemma 1. *With X and Y fixed, if we change Z by allowing it to grow without bound, then z/Z approaches one in the limit.*

Proof. By Lemma 3 in (Rieck, 2014), the quantity $1-c_1^2-c_2^2-c_3^2+2c_1c_2c_3$ equals

$$\frac{(x_1y_2+x_2y_3+x_3y_1-x_1y_3-x_2y_1-x_3y_2)^2 z^2}{\prod_{j=1}^3 [(x-x_j)^2+(y-y_j)^2+z^2]},$$

and also equals

$$\frac{(x_1y_2+x_2y_3+x_3y_1-x_1y_3-x_2y_1-x_3y_2)^2 Z^2}{\prod_{j=1}^3 [(X-x_j)^2+(Y-y_j)^2+Z^2]}.$$

As Z increases, this tends to zero. Therefore z must also increase without bound, or else z must also tend to zero. We need to discount the latter possibility.

Observe that $1-c_1^2$ equals

$$\frac{[(y_3-y_2)x+(x_2-x_3)y+(x_3y_2-x_2y_3)]^2+[(x_2-x_3)^2+(y_2-y_3)^2]z^2}{\prod_{j=2}^3 [(x-x_j)^2+(y-y_j)^2+z^2]},$$

and also equals

$$\frac{[(y_3-y_2)x+(x_2-x_3)y+(x_3y_2-x_2y_3)]^2+[(x_2-x_3)^2+(y_2-y_3)^2]Z^2}{\prod_{j=2}^3 [(X-x_j)^2+(Y-y_j)^2+Z^2]}.$$

To see this, first solve (1) for c_1, and then check that $1-c_1^2=-([(r_2-r_3)^2-d_1^2][(r_2+r_3)^2-d_1^2])/(4r_2^2r_3^2)$ $=(2r_2^2r_3^2+2d_1^2r_2^2+2d_1^2r_3^2-r_2^4-r_3^4-d_1^4)/(4r_2^2r_3^2)$. Then make further substitutions and simplify.

As $Z\to\infty$, $1-c_1^2$ tends to zero. Now suppose $z\to 0$ too. This forces $(y_3-y_2)x+(x_2-x_3)y+(x_3y_2-x_2y_3)=0$. That is, the point (x,y) in the xy-plane must be on the line that connects the control points P_2 and P_3. By similar reasoning concerning c_2 and c_3, (x,y) must also be on the line connecting P_3 and P_1, as well as the line connecting P_1 and P_2. This is impossible. So z cannot be approaching 0 as $Z\to\infty$.

So z must instead increase without bound. Now notice that $1-c_1^2-c_2^2-c_3^2+2c_1c_2c_3$ is asymptotic to $(x_1y_2+x_2y_3+x_3y_1-x_1y_3-x_2y_1-x_3y_2)^2z^{-4}$ and also to $(x_1y_2+x_2y_3+x_3y_1-x_1y_3-x_2y_1-x_3y_2)^2Z^{-4}$. This means that z and Z are asymptotic too, *i.e.* we have $z/Z\to 1$.

□

3.2 the main result

Theorem 1 of (Rieck, 2014) asserts that a certain quantity that depends only on the parameters d_1, d_2, d_3, c_1, c_2, and c_3, also equals

$$A(\phi_1,\phi_2,\phi_3;x,y)+B(\phi_1,\phi_2,\phi_3;x,y)(1-x^2-y^2)/z^2, \tag{3}$$

where $A(\phi_1,\phi_2,\phi_3;x,y)$ and $B(\phi_1,\phi_2,\phi_3;x,y)$ are certain quadratic polynomials in x and y. This formula has two parts, an "A part" and a "B part." Notice that the latter involves the factor $(1-x^2-y^2)/z^2$, which goes to zero as $|z|$ goes to infinity while fixing x and y.

Here we can of course replace (x,y,z) with (X,Y,Z). Fixing (X,Y), as we let $Z\to\infty$, the above quantity approaches $A(\phi_1,\phi_2,\phi_3;X,Y)$. By Lemma 1, $z\to\infty$ too, so the same quantity also approaching $A(\phi_1,\phi_2,\phi_3;x,y)$. For large Z, we thus have

$$A(\phi_1,\phi_2,\phi_3;x,y)\approx A(\phi_1,\phi_2,\phi_3;X,Y), \tag{4}$$

where "≈" becomes equality in the limit.

One further assumption will be quite useful, though it was not used in (Rieck, 2014). Henceforth, we will assume that $\phi_1+\phi_2+\phi_3=0$. This imposes no serious constraint since our chosen coordinate system can always be rotated to make this so. This substantially simplifies the equations and associated geometric interpretation to follow.

Lemma 2. *Assuming that $\phi_1+\phi_2+\phi_3=0$, the equation*

$$A(\phi_1,\phi_2,\phi_3;x,y)=A(\phi_1,\phi_2,\phi_3;X,Y)$$

becomes simply

$$2\,(1+x_3)\,(y+xy-Y-XY) = y_3\,(x^2-2x-y^2-X^2+2X+Y^2). \tag{5}$$

Proof. Using Theorem 1 and the last part of Lemma 3 in (Rieck, 2014), and after making the substitution

$$t_3 \to (t_1+t_2)/(t_1t_2-1)$$

(because $\phi_3 = -\phi_1-\phi_2$), and simplifying, one obtains $A(\phi_1,\phi_2,\phi_3;x,y) =$

$$\frac{(t_1+t_2)(3t_1^2t_2^2-t_1^2-t_2^2-8t_1t_2+3)}{(t_2-t_1)(1+t_1^2)(1+t_2^2)} + \frac{(t_1+t_2)(y^2-x^2+2x)+2(t_1t_2-1)(x+1)y}{t_2-t_1}.$$

This remains so when (x,y) is replaced with (X,Y). The difference between these equals

$$\frac{(t_1+t_2)(y^2-x^2+2x-Y^2+X^2-2X)}{t_2-t_1} + \frac{2(t_1t_2-1)(xy+y-XY-Y)}{t_2-t_1}.$$

This equals

$$\frac{t_3}{x_1-x_2}\,\left[\,y_3\,(y^2-x^2+2x-Y^2+X^2-2X) + 2\,(1+x_3)\,(xy+y-XY-Y)\,\right],$$

which follow by substituting $x_3 \to (1-t_3^2)/(1+t_3^2)$ and $y_3 \to 2t_3/(1+t_3^2)$, and then applying the above substitution for t_3. This yields the claim in the lemma. Note that $t_3/(x_1-x_2) = (1+t_1^2)(1+t_2^2)\,/\,[2(t_2-t_1)(t_1t_2-1)]$, which cannot equal zero, though it is undefined if $\phi_3 = 0$. However, the $\phi_3 = 0$ case can also be accounted for, by continuity, allowing ϕ_1, ϕ_2 and ϕ_3 to vary in such a way that $\phi_1+\phi_2+\phi_3 = 0$ throughout. □

Lemma 3. *Assume that $\phi_1+\phi_2+\phi_3 = 0$. In the limiting case where $Z \to \infty$ (holding X and Y fixed), equations (5) through (9) all hold. Additionally, if (x,y) and (X,Y) are distinct points, then (10) and (11) hold as well.*

Proof. By symmetry, equation (5) gives rise to two other similar equations, as follows:

$$2\,(1+x_1)\,(y+xy-Y-XY) = y_1\,(x^2-2x-y^2-X^2+2X+Y^2) \tag{6}$$

and

$$2\,(1+x_2)\,(y+xy-Y-XY) = y_2\,(x^2-2x-y^2-X^2+2X+Y^2). \tag{7}$$

Taking a linear combination of these yields

$$(y_1+x_2y_1-y_2-x_1y_2)(y+xy-Y-XY) = 0.$$

But

$$y_1+x_2y_1-y_2-x_1y_2 = \frac{4(t_1-t_2)}{(1+t_1^2)(1+t_2^2)} \neq 0,$$

and hence we must conclude that

$$(x+1)y \;=\; (X+1)Y. \tag{8}$$

Then from (5), (6) and (7), we can also deduce that

$$(x-1)^2-y^2 \;=\; (X-1)^2-Y^2. \tag{9}$$

Eliminating y from (8) and (9) yields the resultant polynomial $(x-X)\cdot$

$$[x^3+Xx^2+(Y^2+2X-3)x+(XY^2+2Y^2+X-2)].$$

This must vanish, and so if $x \neq X$, we obtain

$$x^3+Xx^2+(Y^2+2X-3)x+(XY^2+2Y^2+X-2) = 0. \tag{10}$$

By instead eliminating x from (8) and (9), and assuming $y \neq Y$, we likewise obtain

$$y^3+Yy^2+(X+1)(X-3)y+(X+1)^2Y = 0. \tag{11}$$

Concerning the claim in the lemma about (10) and (11), consider first the case where $X \neq -1$ and $Y \neq 0$. Equation (8) here implies that $X = x$ if and only if $Y = y$. So under the assumption that (X,Y) and (x,y) are distinct, (10) and (11) both hold.

Next, consider the two special cases where $X = \pm 1$ and $Y = 0$. Here (10) can be seen to hold if $x = X$, and as has already been observed, it must hold when $x \neq X$. Likewise (11) must hold whether or not $y = Y$.

Lastly, consider the case where $X = -1$ or $Y = 0$, but $X^2+Y^2 \neq 1$, and again assume that (X,Y) and (x,y) are distinct. We can consider making infinitesimal changes to c_1, c_2 and c_3, causing corresponding infinitesimal changes to (X,Y,Z) and (x,y,z). Because (X,Y,Z) is not on the danger cylinder, the mapping between (c_1,c_2,c_3) and (X,Y,Z) is locally invertible (cf. (Rieck, 2014) and (Zhang and Hu, 2006)). Thus the infinitesimal changes can be made so as to cause $X \neq -1$ and $Y \neq 0$, and it will keep (X,Y) and (x,y) distinct. By the first case, (10) and (11) hold for these new points, but then by continuity, (10) and (11) must also hold for the original points.

The above three cases cover all the possibilities for which (X,Y) and (x,y) are distinct points. So the claim stated in the lemma concerning (10) and (11) is true.

□

Notice that (8) and (9) mean that the reference point and all of its related solution points project onto the xy-plane at the points of intersection of two rectangular hyperbolas.

We now arrive at the principal result of this article. Given a reference point (X,Y,Z), we are interested in determining the related points (x,y,z), in a manner that draws a clear connection between (x,y,z) and (X,Y,Z). These points satisfy the combined system (1) and (2) for the same values of the parameters c_1,c_2,c_3,d_1,d_2,d_3. These points need not be real-values, in general. This goal was accomplished for a very simple example in Subsection 2.2, where the reference point was $(0,0,Z)$.

Concerning the problem more generally, this article manages to fulfill the stated goal in the limiting case where $|1-X^2-Y^2|/Z^2$ is infinitesimally small. However, this provides a good sense of how the various solutions to (1) and (2) are related when $|1-X^2-Y^2|/Z^2$ is finite but small, though admittedly, in this case, Theorem 1 only provides approximations to each of the solution points related to a specified reference point.

Theorem 1. *Assume that* $\phi_1+\phi_2+\phi_3=0$. *Given a reference solution point* (X,Y,Z)*, let* $\Delta =$

$$27-24XY^2+8X^3-18(X^2+Y^2)-(X^2+Y^2)^2.$$

Let $\sqrt{-3\Delta}$ *be either of the (complex) square roots of* -3Δ. *Let*

$$\Gamma \;=\; (X-3)^3+9(X+3)Y^2 \;+\; 3\sqrt{-3\Delta}\,Y\,.$$

Let $\sqrt[3]{\Gamma}$ *be any of the three (complex) cube roots of* Γ. *Take*

$$x_0 \;=\; -\frac{1}{3}\left[X+\sqrt[3]{\Gamma}+\frac{(X-3)^2-3Y^2}{\sqrt[3]{\Gamma}}\right]$$

and

$$y_0 \;=\; \frac{(X+1)Y}{x+1}\;.$$

Now, letting $Z\to\infty$*, one of the related solution points* (x,y,z) *for* (X,Y,Z) *is such that* $x\to x_0$ *and* $y\to y_0$*. Conversely, the first two coordinates of each related solution point, in the limit, can be obtained in this manner.*

Proof. The left side of (10) is a cubic polynomial in x, x^3+bx^2+cx+d, with $b=X$, $c=Y^2+2X-3$ and $d=XY^2+2Y^2+X-2$. Following a classical technique for finding its roots, set $\Delta_0=b^2-3ac=(X-3)^2-3Y^2$, and set $\Delta_1=2b^3-9bc+27d=2[(X-3)^3+9(X+3)Y^2]$. Now, $\Delta_1^2-4\Delta_0^3=-108Y^2\Delta$. So $\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}=2\Gamma$. The roots of the cubic are thus $-(b+\sqrt[3]{\Gamma}+\Delta_0/\sqrt[3]{\Gamma})/3$, by a well-known formula for the roots of cubics (essentially Cardano's formula). The roots are the first coordinates of the related solution point. The matching second coordinates can be obtained using (8). □

After using Theorem 1 to solve for x and y, the remaining coordinate z can be determined from the systems (1) and (2), as follows. Using (2), each r_j^2 can be written as a linear function of z^2 with known coefficients. Using (1), each c_j^2 is equal to a rational function whose numerator and denominator are known quadratic functions of z^2. Since the values of the c_j are known, this leads to three quadratic equations in z^2 with known coefficients. Taking any two of these, one can then reduce these to a linear equation in z^2 with known coefficients, which can then be solved for z^2.

Corollary 1. *Assume the reference point* P *has real coordinates* (X,Y,Z)*. Given a related solution point* (x,y,z)*, let*

$$\delta = 27-24xy^2+8x^3-18(x^2+y^2)-(x^2+y^2)^2.$$

In the limiting case where $Z\to\infty$ *(with* X *and* Y *fixed), there are three possibilities:*

1. *If* $\Delta>0$*, then* P *has three distinct real-valued related solution points, and each of these satisfies* $\delta>0$.
2. *If* $\Delta=0$*, then* P *has three real-valued related solution points, and each of these satisfies* $\delta\geq 0$*, but at least two of them coalesce to form a repeated solution point.*
3. *If* $\Delta<0$*, then* P *has exactly one real-valued related solution point, and it satisfies* $\delta<0$.

Proof. As a cubic polynomial in x, the discriminant of (10) is $4Y^2\Delta$. Likewise, as a cubic polynomial in y, the discriminant of (11) is $4(X+1)^2\Delta$. Since the discriminant of (10) is $4Y^2\Delta$, it has three district real roots when $\Delta>0$. If $\Delta=0$, it still has only real roots but at least two of these coalesce, resulting in at most

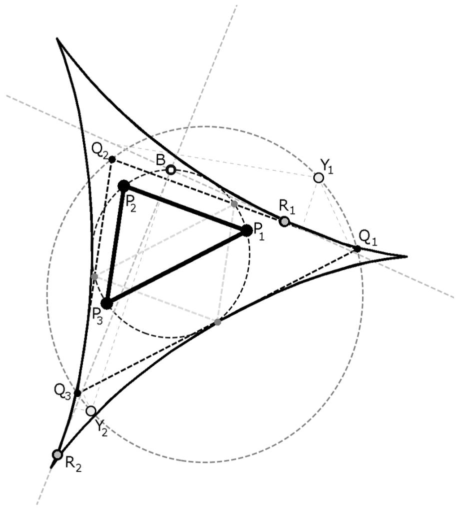

Figure 2: the deltoid and related geometry

two distinct roots. If $\Delta < 0$, (10) has only one real root. For each root of (10), there is a matching root of (11) that can be found by simply using (8). □

3.3 the deltoid

The equation $\Delta = 0$ describes a classical curve known as a "deltoid." This same curve has also been encountered in studies of "triangle geometry" (Zwikker, 1963). While, as yet, there is no intelligible connection between the historic appearance of deltoids from triangles with the present appearance of deltoids in the P3P problem, it seems sensible to outline the former here, and to point towards such a connection with the P3P problem.

In Figure 2[1] the thick triangle $\Delta P_1P_2P_3$ is the starting point for this discussion, and its vertices will turn out to correspond to the control points of the P3P problem. This triangle is reflected in its circumcircle, producing the light dashed triangle. From this, the triangle $\Delta Q_1Q_2Q_3$ is constructed so that the light dashed triangle is the medial triangle of $\Delta Q_1Q_2Q_3$ (so $\Delta Q_1Q_2Q_3$ is the anti-medial triangle of the light dashed triangle). Observe that $\Delta P_1P_2P_3$ and $\Delta Q_1Q_2Q_3$ are similar triangles.

Two antipodal points Y_1 and Y_2 are placed arbitrarily on the circumcircle of $\Delta Q_1Q_2Q_3$. The orthogonal projections of Y_1 onto the three lines that extend

[1]Based on the author's animated GIF image, which was inspired by Zachary Abel's animated GIF images at `http://blog.zacharyabel.com/2012/04/three-cornered-deltoids/`.

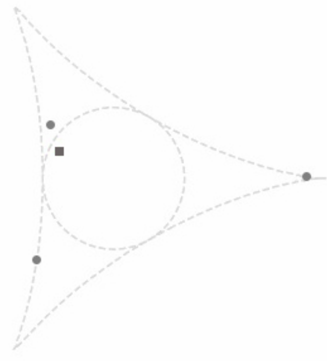
Figure 3: related points in limiting case

the sides of $\Delta Q_1Q_2Q_3$ result in three collinear points. The line containing these points is called a "Simson line." Likewise, the orthogonal projections of Y_2 also yield another Simson line. In fact, it is further known that these two Simson lines are perpendicular to each other. Their point of intersection is denoted B in Figure 2.

Now, as Y_1 and Y_2 are rotated together about the circumcircle for $\Delta Q_1Q_2Q_3$, each of these points produces a pencil of Simson lines, whose envelope is a deltoid. In fact, Y_1 and Y_2 produce the same deltoid. This deltoid in Figure 2 is the large curvy triangle. The point R_1 (R_2) is the point of intersection of Simson lines produced as Y_1 (Y_2) moves an infinitesimal amount. R_1 and R_2 are on the deltoid.

The deltoid thus produced is known as the "Steiner deltoid" (for $\Delta Q_1Q_2Q_3$), and it agrees with the one that occurs in the analysis of the P3P problem. Moreover, there seems to be an even stronger connection with P3P, though there is no apparent explanation for this connection, as yet. It seems that the points B, R_1 and R_2 are the projections onto the xy-plane of the three points considered in Case 2 of the conjecture to be presented next.

3.4 a conjecture

Returning now to the limiting-case P3P problem of Theorem 1, visual simulations exploring various positions of the reference point and the corresponding related points have revealed some interesting behaviors. For instance, if the reference point is inside the danger cylinder, then there are three related points, and it seems that all of these stay inside the deltoidal region. As the reference point is moved close to the danger cylinder, one of the three related points moves towards it (ultimately coalescing with it on the danger cylinder), while the other two move so that their projections onto the *xy*-plane, approach the deltoid. This can be seen in Figure 3, where the square dot is the projection of the reference point onto the *xy*-plane.

The observations made from the various simulations strongly suggests the following.

Conjecture 1. *Assume the reference point P has real coordinates* (X,Y,Z). *There are the following five possibilities:*

1. *If* $X^2+Y^2<1$*, then there are three real-valued related solution points* (x,y,z)*, each with* $x^2+y^2>1$ *and* $\delta>0$.
2. *If* $X^2+Y^2=1$*, then there are three real-valued related solution points, but at least one of these coalesces with the reference point, and except when* (X,Y) *is one of three particular points, exactly one related solution point does so, and the other two satisfy* $\delta=0$.
3. *If* $X^2+Y^2>1$ *and* $\Delta>0$*, then there are three real-valued related solution points, each satisfying* $\delta>0$*. One of these satisfies* $x^2+y^2<1$ *and the other two satisfy* $x^2+y^2>1$.
4. *If* $X^2+Y^2>1$ *and* $\Delta=0$*, then there are three real-valued related solution points, except that at least two of them coalesce and satisfy* $x^2+y^2=1$*. If exactly two of them coalesce, then the other one satisfies* $x^2+y^2>1$ *and* $\delta=0$.
5. *If* $\Delta<0$*, then there is only one real-valued related solution point, and it satisfies* $\delta<0$.

4 CONCLUSION

Together, Theorem 1, Corollary 1 and Conjecture 1 gives a very complete description of the relationship between the solution points for the limiting case examined here. The results in this article provide useful insights that can be carried back to gain a greater understanding of the general situation for P3P. Some simulations have demonstrated that as long as the reference point avoids getting close to the control points plane, the distribution of the related points will be similar to that of the limiting case.

In continuing the analysis of this article, the next step would seem to be to remove the restriction that $|1-X^2-Y^2|/Z^2$ be negligible. This would require the inclusion of the "B part" of the formula in Theorem 1 of (Rieck, 2014). There is little doubt that this would result in significantly more complicated relationships among the solution points. There is however a reasonable hope that some compelling interplay between the solution points will be discovered. Ideally, eventually, a good geometric understanding of all the salient aspects of P3P will emerge.

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