Bundle Adjustment Methods in Engineering Photogrammetry

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Stuart Granshaw
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BUNDLE ADJUSTMENT METHODS IN ENGINEERING PHOTOGRAMMETRY

By S. I. GRANSHAW†
University College London

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Abstract

The bundle method is a flexible analytical tool for co-ordinating engineering and industrial structures. This paper investigates, mainly from a theoretical standpoint, what object space control, if any, need be incorporated in a bundle adjustment and it promotes the use of multistation photography for such projects. The concepts of inner accuracy and free network adjustments are introduced as being valuable in comparative precision studies. Self calibration techniques for the compensation of systematic errors, as well as the detection of gross errors, are also discussed.

INTRODUCTION

IF PHOTOGRAMMETRY is to be used as an aid to the measurement of an engineering or industrial structure, be it a supertanker, a tunnel, a vehicle component, or any other object requiring quantitative analysis, often the need is for a flexible system to obtain point information of high accuracy, which demonstrates the potential advantage of analytical methods over an analogue approach. Several papers have been published in recent years which describe particular applications of analytical photogrammetric measurements to specific engineering and industrial structures (Newton, 1975; Bopp *et al.*, 1977; Kenefick, 1977; Scott, 1978; Cooper, 1979) and it is not intended to augment these. Rather, the means of assessing and achieving the desired accuracy and flexibility required in the co-ordination of such structures will be discussed.

Close range applications of photogrammetry are often considered as requiring mere adaptations of techniques developed specifically for aerial survey. Whilst this approach may be valid in such fields as architectural photogrammetry, we should not be restricted by our widespread experience in conventional stereophotogrammetry when trying to optimise results in applications where something rather different may be required. Although the essential differences between the photogrammetric measurement of an engineering structure and an aerial survey may be self evident, some of these differences have been outlined explicitly in Table I.

Having decided upon an analytical approach, why should the bundle method be preferred to any other analytical technique? The answer lies in the necessity for *flexibility* to achieve the highest accuracies economically, because requirements vary considerably from project to project. What are the possible alternative methods that could be used? On the one hand we could adopt a solution involving the formation of models, followed by the joining of these models as in independent model triangulation. This approach is exemplified by a conventional analytical relative orientation based upon the coplanarity condition as utilised, for example, by Cooper (1979). However, this is to assume that a two photograph relative orientation is "good"‡, as control points can only be introduced into

† Now at Oxford Polytechnic.

‡ There is no doubt that conventional relative orientation procedures produce satisfactory results with aerial photography, but the same may not necessarily be true in close range applications (Granshaw, 1979).

TABLE I. Some differences between engineering photogrammetry and an aerial survey.

Engineering photogrammetry	*Aerial survey*
Object may have truly spatial characteristics (great depth)	Relief is small in comparison with the flying height
Accuracy in all three co-ordinates may be equally important	Accuracy requirements in height different from those in plan
A restricted format is likely	Entire format is usable
Spatial nature of object necessitates photographs with varying position and orientation	Vertical photography employed almost exclusively
Point information is the essential requirement	Photographs may be used for plotting as well as point determination
Targetting may be possible for all points which require co-ordination	Excluding cadastral surveys, targets may only be used for control points, if at all
Total number of photographs is relatively small	A large block may consist of thousands of photographs
Site restrictions frequently encountered	Few restrictions other than air traffic regulations
Possible to use glass plates rather than film	Film has to be used
May be possible to determine some of the camera parameters with considerable accuracy	Auxiliary data have only a limited accuracy
Flexible approach required due to differences from one project to another	A fairly standardised approach can be adopted

the adjustment *after* the formation of the model. On the other hand we could adopt a solution whereby control points are simultaneously incorporated in the adjustment. An example of such a method is the direct linear transformation, first presented by Abdel-Aziz and Karara (1971) and placed on a more rigorous mathematical basis by Bopp and Krauss (1978). However, at least when using metric cameras, we should question whether the inclusion of widespread object space control is really necessary.

The flexibility of the bundle method allows it to encompass both of these, and other, situations. It therefore seems particularly well suited to engineering applications which require the highest accuracies. This flexibility is particularly valuable when carrying out *a priori* accuracy studies (network design). Consider, for example, a hypothetical object which is nominally a cube (Fig. 1). Targets have been placed on its surface and we are

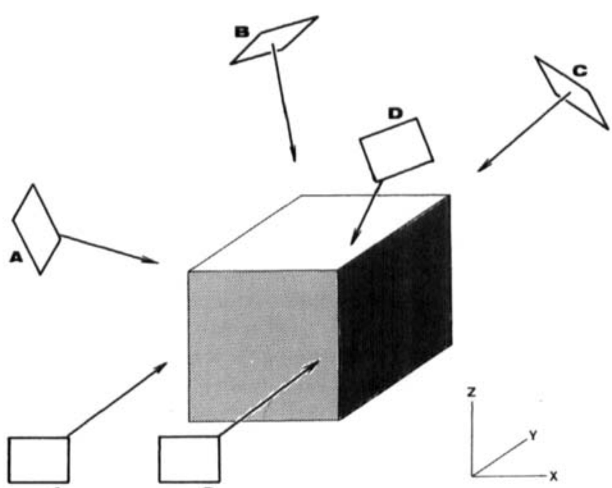

FIG. 1. The co-ordination of a solid cube using multistation photography (A, B, C, D) or conventional stereopairs (1 and 2).

required to co-ordinate these targets in some system. We might take conventional stereopairs of each side in turn, such as pictures 1 and 2, but could a better result be achieved using multistation convergent photography such as A, B, C and D? Would it have helped if a proportion of the targets had been pre-surveyed and these values incorporated simultaneously in the adjustment, or is this unnecessary? These questions serve to indicate two of the main aims of this paper:

(i) to illustrate how the use of multistation photography can improve the accuracy of the final result; and

(ii) to indicate the situations in which the use of object space control produces a significant improvement in the results, and those situations where the utilisation of extensive control is unwarranted.

Multistation photogrammetry may be defined as the measurement of an object where all (or at least, most) of the regions of interest image on three or more spatially separated photographs. This is to be distinguished from conventional stereophotogrammetry where, although certain regions of the object may image on more than two photographs for control and aerial triangulation purposes, the measurement of detail is mainly a function of stereopairs. The bundle method is particularly suited to accommodating multistation photography with, or without, object space measurements. Multistation photogrammetry is probably only really practical when co-ordinating targetted points, but targetting should be considered an integral part of any method where the highest accuracies are required.

In any assessment of methods on the basis of *accuracy*, our interpretation of this term is of fundamental importance. Accuracy is concerned with errors. The classical division of errors into random, systematic and gross errors is a useful one, and will be used as the basis of three sections which follow. Before that, however, we must discuss the bundle adjustment in broad terms to see how it can be applied to engineering measurement problems.

Basic Bundle Solution

The formulation of the bundle solution is briefly reviewed, having regard to its application to the measurement of engineering structures. For the moment, let us consider the ith object point which images on a particular picture (Fig. 2). Let x_i, y_i be the surface

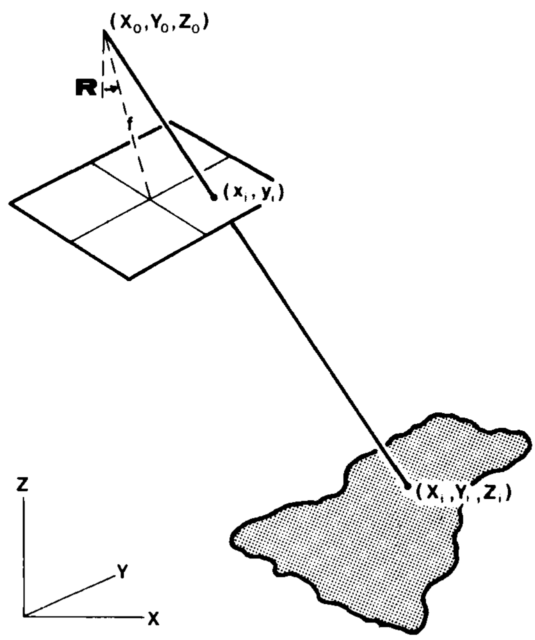

Fig. 2. The geometrical basis of analytical photogrammetry.

co-ordinates of the image, with the principal point as origin, and let the principal distance be f. Let X_i, Y_i, Z_i be the space co-ordinates of the point, let X_0, Y_0, Z_0 be the space co-ordinates of the perspective centre and let r_{ij} be a typical element of an orthogonal matrix **R** which defines the rotation of the picture with respect to the space system. The collinearity equations which relate the surface and space co-ordinates then read:

$$x_i = f\frac{(X_i - X_0)r_{11} + (Y_i - Y_0)r_{21} + (Z_i - Z_0)r_{31}}{(X_i - X_0)r_{13} + (Y_i - Y_0)r_{23} + (Z_i - Z_0)r_{33}}, \tag{1a}$$

$$y_i = f\frac{(X_i - X_0)r_{12} + (Y_i - Y_0)r_{22} + (Z_i - Z_0)r_{32}}{(X_i - X_0)r_{13} + (Y_i - Y_0)r_{23} + (Z_i - Z_0)r_{33}}. \tag{1b}$$

A practical solution of equations (1) requires them to be linearised and as part of this process we choose to let

$$\mathbf{R} = \Delta\mathbf{R}.\mathbf{R}' \tag{2}$$

where $\mathbf{R}'$ is a strictly orthogonal matrix representing the rotations determined by the iterative solution thus far, and $\Delta\mathbf{R}$ is given by the first order form

$$\Delta\mathbf{R} = \begin{pmatrix} 1 & -\theta_Z & \theta_Y \\ \theta_Z & 1 & -\theta_X \\ -\theta_Y & \theta_X & 1 \end{pmatrix}, \tag{3}$$

where $\theta_X, \theta_Y, \theta_Z$ are small rotations about axes through the perspective centre parallel to the co-ordinate axes of the space system. We note that the form of $\Delta\mathbf{R}$ is independent of whether $\mathbf{R}'$ has been determined using ω, φ, κ rotations in some sequence, or using the Rodrigues matrix. The reason for this particular formulation of the rotation matrix will become evident later.

Applying Newton's method to equations (1), the linearised observation equations have a simple form which is given in Appendix A. From these linearised observation equations we can form the normal equations. An example of the structure of the symmetric coefficient matrix of the normal equations for 70 object points and eight pictures is given in Fig. 3(a). Let the partitioned normal equations be given by

$$\begin{pmatrix} \mathbf{N}_p & \mathbf{N}_{ps}^{\mathrm{T}} \\ \mathbf{N}_{ps} & \mathbf{N}_s \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x}_p \\ \Delta\mathbf{x}_s \end{pmatrix} = \begin{pmatrix} \mathbf{t}_p \\ \mathbf{t}_s \end{pmatrix}, \tag{4}$$

where the suffix p indicates object point co-ordinates and the suffix s indicates the camera parameters. Thus $\Delta\mathbf{x}_p$ and $\Delta\mathbf{x}_s$ represent corrections to the object co-ordinates and camera parameters respectively, and $\mathbf{t}_p$ and $\mathbf{t}_s$ are the corresponding right hand sides of the normal equations. If there are n_p object points and n_s camera stations, the entire coefficient matrix illustrated in Fig. 3(a) is of order $(3n_p + 6n_s) \times (3n_p + 6n_s)$. However, the submatrix $\mathbf{N}_p$ is easily inverted and this allows the formation of the following system of *reduced normal equations* (Brown, 1976):

$$\mathbf{N}_r \Delta\mathbf{x}_s = \mathbf{t}_r, \tag{5}$$

where

$$\mathbf{N}_r = \mathbf{N}_s - \mathbf{N}_{ps}\mathbf{N}_p^{-1}\mathbf{N}_{ps}^{\mathrm{T}} \tag{6a}$$

and

$$\mathbf{t}_r = \mathbf{t}_s - \mathbf{N}_{ps}\mathbf{N}_p^{-1}\mathbf{t}_p. \tag{6b}$$

$\mathbf{N}_r$ is of order $6n_s \times 6n_s$ (Fig. 3(c)) and equation (5) can be solved directly for $\Delta\mathbf{x}_s$. Then $\Delta\mathbf{x}_p$ can be determined from

$$\Delta\mathbf{x}_p = \mathbf{N}_p^{-1}(\mathbf{t}_p - \mathbf{N}_{ps}^{\mathrm{T}}\Delta\mathbf{x}_s). \tag{7}$$

The inclusion of observed co-ordinates (control points) and camera parameters is best done using a technique such as the unified method of least squares (Mikhail, 1976)

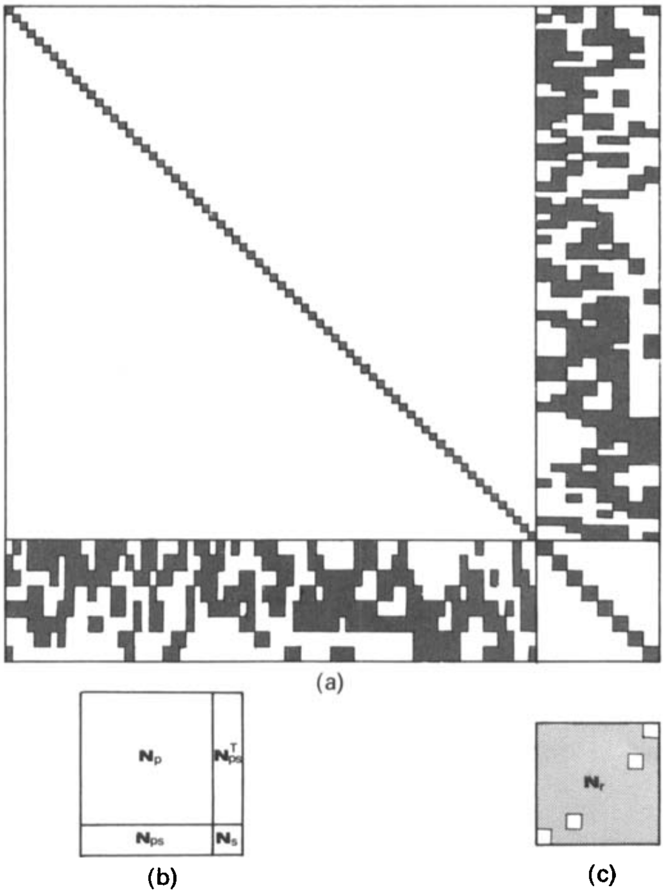

FIG. 3. (a) Structure of the coefficient matrix of the normal equations for 70 object points and eight pictures; (b) notation for the partitioned matrix given in (a); (c) the structure of the corresponding coefficient matrix of the reduced normal equations. Zero elements are left blank.

where, conceptually, all parameters are treated as weighted observations. Using this method, the structure of the coefficient matrix of the normal equations is not modified from that given in Fig. 3(a) provided that the dispersion (variance–covariance) matrices of the non-photogrammetric observations are compatible with the form illustrated in Fig. 3(a). The inclusion of n_d distances between object points, which may be of great value in engineering photogrammetry, necessitates the inversion of n_d 6×6 submatrices in addition to the usual inversion of a series of 3×3 submatrices to form $\mathbf{N}_p^{-1}$. However, the inclusion of further observations in the object space, including the forms of lines and regular surfaces, really requires the partitioning of $\mathbf{N}_p$ (Hell, 1979) and is not considered further in the present paper. We note that if an object point does not image on a particular photograph, the corresponding 6×3 submatrix of $\mathbf{N}_{ps}$ is null and advantage can be taken of this by packing $\mathbf{N}_{ps}$. However, in many engineering applications the structure of $\mathbf{N}_{ps}$ will not be particularly regular (in contrast to aerial photography), with the result that the coefficient matrix of the reduced normal equations, $\mathbf{N}_r$, is virtually full in many cases. (Fig. 3(c) illustrates the structure of the coefficient matrix of the reduced normal equations for the partitioned matrix of the full normal equations shown in Fig. 3(a). The i,jth 6×6 submatrix of $\mathbf{N}_r$ will be null only if none of the points imaging on the ith picture also image on the jth picture.) Thus the use of special procedures for the solution of the reduced normal equations, which are essential with large blocks of aerial triangulation where $\mathbf{N}_r$ has a banded structure (Brown, 1968, 1976), are of little value and a direct Cholesky or

Gauss–Doolittle solution can be adopted in the rather small blocks encountered with engineering structures (Mayoud, 1978).

The bundle solution, formulated using the unified method of least squares, has great flexibility. By a suitable choice of weight matrices we can perform a space resection, a conventional relative orientation, a multistation relative orientation or a simultaneous relative and absolute orientation, as well as the conventional bundle adjustment, although it may not, of course, be the most efficient means of achieving all of these objectives.

Redundancy

As one of the purposes of this paper is to draw attention to multistation methods, it may be instructive to look briefly at the question of redundancy in a multistation relative orientation (that is, the orientation of three or more photographs relative to one another).

In a conventional two photograph relative orientation we require a minimum of five object points to image on both photographs and each additional point adds one more redundancy to the solution. In a collinearity solution of a multistation relative orientation (thus using a bundle approach), if n_p object points all image on n_s pictures, we will have $2n_p n_s$ observations and $(3n_p+6n_s-7)$ unknowns. Thus r, the redundancy, is given by

$$r = 2n_p n_s-(3n_p+6n_s-7)$$

or

$$r = (n_p-3)(2n_s-3)-2 \geqslant 0. \qquad (8)$$

A graph of redundancy against the number of object points for different numbers of pictures is given in Fig. 4. From this graph it can be seen that the redundancy increases

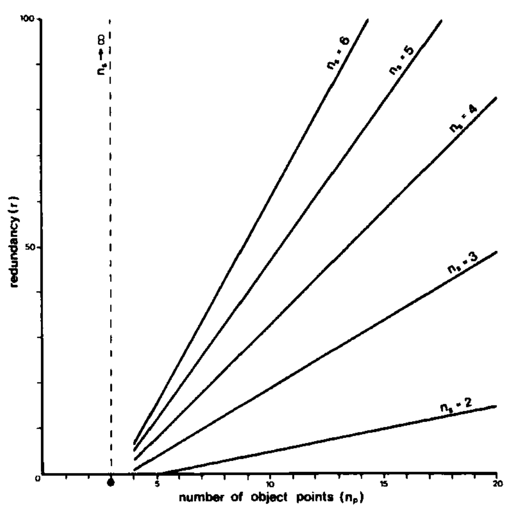

FIG. 4. Redundancy in a multistation relative orientation. It is assumed that all object points image on all photographs. n_s is the number of photographs.

rapidly as the number of photographs is increased. An interesting point concerns the intersection of the lines with the abscissa ($r = 0$). Fig. 4 seems to suggest that, whereas a minimum of five points is required for a two photograph relative orientation, four is the minimum for a general multistation relative orientation ($n_s \geqslant 3$). For a zero redundancy we have, from equation (8),

$$n_p = \frac{2}{2n_s - 3} + 3. \tag{9}$$

In the limit, as $n_s \to \infty$, $n_p \to 3$, and so we must always have at least four points, the use of three being reserved for solutions involving known object space co-ordinates (a space resection, for example). However, such redundancy figures only indicate necessary and not sufficient conditions. Indeed, failure cases in a four point multistation relative orientation seem to be far more common than in a conventional five point orientation, but Fig. 5 illustrates a situation in which a three picture relative orientation produces a strong solution using four points. Such facts may be of purely academic interest, but they do underline that one must look at engineering structures, where multistation geometry may be relevant, in a manner which is not too myopic due to the shackles of aerial mapping.

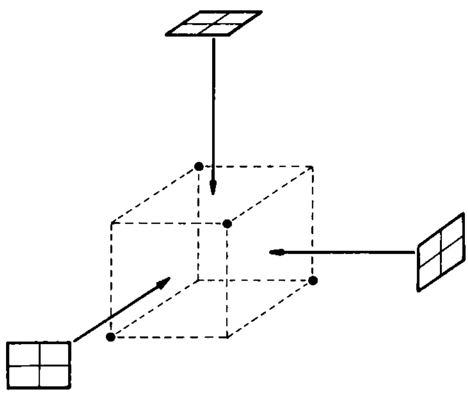

FIG. 5. A situation where four points imaging on three photographs produce a strong multistation relative orientation.

Theoretically, an improvement in precision could be achieved by taking additional exposures at the same camera stations, thereby increasing the redundancy, as has been suggested, for example, by Hottier (1976). In practice there are two reasons why the use of multistation geometry (where the additional exposures are secured from completely different camera stations) is to be preferred.

(i) Employment of multistation photography will cause the precision to be more homogeneous in all three co-ordinate directions (Kenefick, 1971).

(ii) Significant precision improvement with increased redundancy will only occur if the superfluous observations are relatively uncorrelated (Thompson, 1976). Observations from multistation photography (where the camera stations are widely separated in space) are less likely to be highly correlated than observations from photographs taken at the same stations.

PROPAGATION OF RANDOM ERRORS

Observations may contain three types of errors: random, systematic and gross errors. In this section, we will consider the problem of comparing the accuracy of different bundle solutions when only random errors are present in the observations. In other words, we are concerned with *precision* rather than accuracy as such. Systematic and gross errors will be considered in subsequent sections.

If random errors of a particular magnitude are present in the observed picture co-ordinates, they will affect the precision of the required object co-ordinates in two ways.
(i) If the orientation of the photographs is known and fixed *a priori*, the random errors

will affect the precision of the object co-ordinates in a manner which will depend solely upon the number and geometry of the rays which intersect (nominally) in that point.
(ii) If the orientation of the photographs is unknown, as it usually will be, observations (including picture co-ordinates) are used to determine the orientation of the photographs. This orientation will be imperfect to some degree and will contribute additional errors to the computed object co-ordinates.

On this basis, any examination of precision can also be divided into two parts.
(i) How do changes in the number, position and orientation of the photographs (the geometry of the photogrammetric network) affect the precision of the final co-ordinates?
(ii) How does the inclusion of object space control, including target co-ordinates, distances, perspective centre co-ordinates and camera attitude data, help to improve the orientation? It is clear that the only way in which object space control can increase the precision of the photogrammetrically determined points is by improving the orientation of the photographs.

Precision Estimates for the Bundle Adjustment

An estimate of the precision of the final results may be important in high accuracy engineering applications of photogrammetry and is essential in research where we wish to compare various geometrical and control point configurations. The most universal estimate of precision is the dispersion (variance–covariance) matrix derived from the inverse of the coefficient matrix of the normal equations. Dispersion matrices have been utilised for network analysis in surveying for many years (Allman and Hoar, 1973), but their use in aerial triangulation has been severely limited by the impracticability of computing the inverse, or even parts of it, for large blocks of photography, although certain theoretical investigations have been undertaken (Ebner *et al.*, 1977). However, with the relatively small systems of equations which occur in engineering applications of the bundle adjustment, we can invert all, or part, of the coefficient matrix of the normal equations provided that we follow the partitioning scheme outlined in the previous section. However, this has not always been done in the past. For example, in reference to precision figures quoted for photogrammetrically determined points, Kenefick (1971) states: "the implicit assumption has been made that the relative orientation is known [determined] with such a high degree of precision as to have no detrimental effect on the error propagation". Although this may (or may not) be the case for well designed networks, it cannot be assumed to be true for general engineering applications. Of course, we also make use of the common (although almost certainly fallacious) assumption that the *a priori* dispersion matrix of the photogrammetric observations is a scalar matrix (in other words, the picture co-ordinates are uncorrelated and of equal weight), although at least this assumption has the merit of being common to all the results which we wish to compare.

Let the partitioned inverse of the coefficient matrix of the normal equations (4) be given by

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}_p & \mathbf{C}_{ps}^{\mathrm{T}} \\ \mathbf{C}_{ps} & \mathbf{C}_s \end{pmatrix}. \tag{10a}$$

That is

$$\begin{pmatrix} \mathbf{C}_p & \mathbf{C}_{ps}^{\mathrm{T}} \\ \mathbf{C}_{ps} & \mathbf{C}_s \end{pmatrix} \begin{pmatrix} \mathbf{N}_p & \mathbf{N}_{ps}^{\mathrm{T}} \\ \mathbf{N}_{ps} & \mathbf{N}_s \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{10b}$$

Expansion of equation (10b) leads to the following results (compare with Brown, 1976). $\mathbf{C}_s$ is given directly by inverting the coefficient matrix of the reduced normal equations (5):

$$\mathbf{C}_s = \mathbf{N}_r^{-1}. \tag{11}$$

In addition,

$$\mathbf{C}_{ps} = -\mathbf{C}_s \mathbf{N}_{ps} \mathbf{N}_p^{-1} \tag{12}$$

and

$$\mathbf{C}_p = \mathbf{N}_p^{-1} + (\mathbf{N}_{ps}\mathbf{N}_p^{-1})^{\mathrm{T}}\mathbf{C}_s(\mathbf{N}_{ps}\mathbf{N}_p^{-1}). \tag{13}$$

The precision estimates (variances and covariances) derived from equations (11) to (13) are, of course, relative to the adopted co-ordinate system. This causes no particular problems in a traditional bundle solution because the co-ordinate system is determined by the control. However, photogrammetry is a method of spatial triangulation which means that, without control, it can only determine shape (or size and shape if at least one distance is included). In certain engineering applications this may be all that is required.† Thus the object co-ordinates need not be tied to any particular system and, in such cases, the use of control points would have the sole purpose of strengthening the photogrammetric restitution. Even if a particular co-ordinate system is essential for a given project, it would be advantageous to know whether the control points need be simultaneously incorporated in the bundle adjustment, or whether an absolute orientation could be performed subsequent to a (multistation) relative orientation with no detrimental effect.

To fix our ideas, consider the problem of determining the shape of the nominal cube shown in Fig. 1. As we have noted, its shape can be determined by purely photogrammetric measurement in a multistation relative orientation. We then ask ourselves what is the precision of the determination? To perform the relative orientation we have to fix, arbitrarily, seven parameters in the solution. In a bundle adjustment method, such a choice can be made from the camera station parameters, object co-ordinates, or a combination of the two, and each different selection produces a different dispersion matrix. The problem becomes even more complex if we then ask whether the precision would be significantly improved by incorporating control points into the adjustment. Consider another question. We perform a (multistation) relative orientation and then wish to carry out an absolute orientation. What is the precision of the model co-ordinates derived from the relative orientation? Is the implicit assumption of equal weights valid?

In this connexion, the concepts of free network adjustments and inner accuracy are useful ones, especially when we wish to *compare* the precision to which an object's shape is determined with and without the use of object space control.

Inner Accuracy and Free Network Adjustments

The concept of *inner accuracy*‡, introduced by Meissl (1962), allows the effect of an arbitrary set of parameters to be filtered out of a dispersion matrix. If the filter parameters are selected to be rotational, translational and scaling parameters, then the inner accuracy refers to the internal precision of the network which is not affected by the choice of co-ordinate system. Moreover, when these filter parameters are conceptually identical with arbitrary constraints imposed to allow a solution, the inner accuracy refers to the precision of a *free network adjustment*. To clarify these ideas, consider a bundle adjustment where we are only concerned with the *shape* of an engineering structure. Here the filter parameters are the seven elements of absolute orientation. We may have included object space control in an attempt to improve the solution, but if the shape is determined without control, using a (multistation) relative orientation, we have a free network adjustment.

In a free network adjustment we attempt to determine a solution vector and a dispersion matrix when the system of normal equations is singular due to a rank defect (the rank defect being seven in the case of a relative orientation). Any solution vector will be biased, statistically, but as only the shape defined by the co-ordinates, and not the co-ordinates themselves, is important, this is not of significance. In a free network adjustment we choose a solution where the weighted trace of the dispersion matrix is a minimum. The geometrical interpretation of the minimum trace is that there should be no *overall* translational, rotational or scaling changes from the approximate values. Thus precision

† Note the comments of J. F. Kenefick in "Forum", *Photogrammetric Engineering and Remote Sensing*, 45(9): 1229 (September 1979).

‡ The term "internal precision" may be a better translation of the German *innere Genauigkeit*.

estimates are referred not to particular (arbitrary) points, but to the network of points as a whole. In order to clarify this point, consider a series of practical experiments which attempt to determine the accuracy of model co-ordinates derived from a relative orientation. Photographs are taken of a network of targetted points which have been co-ordinated by suitable measurements to an accuracy much higher than that obtainable by photogrammetric means. The images of these targets are measured and a relative orientation is performed. In order to compare the derived model co-ordinates with the known target co-ordinates, an absolute orientation has to be carried out. The experiment is repeated a thousand times, say. Provided that the absolute orientations have been performed using *all* of the measured targets and that there are no significant systematic or gross errors present in the picture co-ordinates, the standard errors derived from the experiment will be conceptually equivalent with the precision estimate of a free network adjustment.

Despite the fact that one of the earliest applications of a free network adjustment was to the relative orientation of a photogrammetric model (Meissl, 1965), the method has received most attention in connexion with geodetic networks, although we note that the wider concept of inner accuracy has been used by Ebner (1974) and Grün (1976) in a photogrammetric context. The results can be determined by transformations on arbitrary dispersion matrices (Meissl, 1962, 1964), generalised inverses (Mittermayer, 1972a), and zero eigenvalue concepts (Mittermayer, 1972b), amongst others. An alternative derivation is outlined in Appendix B. Other useful summaries are given by Ashkenazi (1974) and Welsch (1979).

Free Bundle Adjustment

If we confine our attention to obtaining a dispersion matrix where only the shape of an object is important, we can summarise the main results of Appendix B in the context of a bundle adjustment.

If the coefficient matrix of the linearised observation equations (1) is **A** (see Appendix A) of order $2n_i \times (3n_p + 6n_s)$, where n_i is the number of image points measured, then we look for a matrix **G** of order $(3n_p + 6n_s) \times 7$ which satisfies the following two conditions.
(i) The columns of **G** are linearly independent, both between themselves and with the columns of **A**.
(ii) $\mathbf{AG} = \mathbf{0}$. (14)
The first condition will be fulfilled if each column of **G** is constructed with the seven absolute orientation parameters in mind. Thus the sum of the translations in each co-ordinate direction should be zero, the sum of the rotations about the co-ordinate axes should be zero, and there should be no overall scale change (see also equation (B.11), Appendix B). Examples of **G** for geodetic networks are given in Meissl (1969) and Mittermayer (1972b). By considering the seven filter parameters (absolute orientation elements) the form of **G** can be extended to the following form for the bundle solution. Partitioning **G** compatibly with equation (4) we have

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_p \\ \mathbf{G}_s \end{pmatrix}, \tag{15}$$

where

$$\mathbf{G}_p^{\mathrm{T}} = \left(\begin{array}{ccc:ccc:c:ccc}
1 & 0 & 0 & 1 & 0 & 0 & \cdots\cdots & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & \cdots\cdots & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & \cdots\cdots & 0 & 0 & 1 \\
0 & -Z_1 & Y_1 & 0 & -Z_2 & Y_2 & \cdots\cdots & 0 & -Z_p & Y_p \\
Z_1 & 0 & -X_1 & Z_2 & 0 & -X_2 & \cdots\cdots & Z_p & 0 & -X_p \\
-Y_1 & X_1 & 0 & -Y_2 & X_2 & 0 & \cdots\cdots & -Y_p & X_p & 0 \\
X_1 & Y_1 & Z_1 & X_2 & Y_2 & Z_2 & \cdots\cdots & X_p & Y_p & Z_p
\end{array}\right) \tag{16}$$

and

$$
\mathbf{G}_s^{\mathrm{T}} = \left(\begin{array}{cccccc:cccccc:c:cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -Z_{0_1} & Y_{0_1} & 1 & 0 & 0 & 0 & -Z_{0_2} & Y_{0_2} & 1 & 0 & 0 & \cdots & 0 & -Z_{0_s} & Y_{0_s} & 1 & 0 & 0 \\
Z_{0_1} & 0 & -X_{0_1} & 0 & 1 & 0 & Z_{0_2} & 0 & -X_{0_2} & 0 & 1 & 0 & \cdots & Z_{0_s} & 0 & -X_{0_s} & 0 & 1 & 0 \\
-Y_{0_1} & X_{0_1} & 0 & 0 & 0 & 1 & -Y_{0_2} & X_{0_2} & 0 & 0 & 0 & 1 & \cdots & -Y_{0_s} & X_{0_s} & 0 & 0 & 0 & 1 \\
X_{0_1} & Y_{0_1} & Z_{0_1} & 0 & 0 & 0 & X_{0_2} & Y_{0_2} & Z_{0_2} & 0 & 0 & 0 & \cdots & X_{0_s} & Y_{0_s} & Z_{0_s} & 0 & 0 & 0
\end{array}\right) \tag{17}
$$

The first three rows of $\mathbf{G}^{\mathrm{T}}$ are associated with the translation parameters, the second three rows with the rotation parameters and the seventh row with the scaling factor. Provided that the rotation matrices of the cameras are in the form given by equations (2) and (3), the form of $\mathbf{G}$ given in equations (15), (16) and (17) satisfies equation (14), as is shown in Appendix C.

Referring to Appendix B, we can consider that, if the normal equations $\mathbf{N}\Delta\mathbf{x} = \mathbf{t}$ are singular due to a free network design, they can be augmented using the matrix $\mathbf{G}$ in equation (15) to form the non-singular system

$$
\begin{pmatrix} \mathbf{N} & \mathbf{PG} \\ \mathbf{G}^{\mathrm{T}}\mathbf{P} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{t} \\ \mathbf{0} \end{pmatrix} \tag{18}
$$

where $\mathbf{P}$ is an arbitrary weight matrix and $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers. The minimum weighted trace dispersion matrix can then be derived by inverting the (non-singular) coefficient matrix in equation (18).

If the system of normal equations is non-singular, but we wish to filter out the effect of the absolute orientation parameters, the dispersion matrix $\mathbf{C}$ derived from the inverse of the non-singular coefficient matrix can be transformed into the minimum weighted trace dispersion matrix $\mathbf{Q}$ using the transformation

$$
\mathbf{Q} = (\mathbf{I} - \mathbf{G}(\mathbf{G}^{\mathrm{T}}\mathbf{PG})^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{P})\mathbf{C}(\mathbf{I} - \mathbf{PG}(\mathbf{G}^{\mathrm{T}}\mathbf{PG})^{-1}\mathbf{G}^{\mathrm{T}}). \tag{19}
$$

This latter form can also be used for a true free network adjustment (for example a relative orientation) where the dispersion matrix $\mathbf{C}$ has been computed by applying arbitrary constraints. The resulting dispersion matrix ($\mathbf{Q}$) will be identical to that obtained by inverting the coefficient matrix in equation (18).

The form of the weight matrix $\mathbf{P}$ in equations (18) and (19) will depend upon the required interpretation of the results. In geodetic free network adjustments this weight matrix is often an identity matrix ($\mathbf{P} = \mathbf{I}$), so that the sum of the variances of all the co-ordinates are minimised. However, in a bundle adjustment the solution involves perspective centre co-ordinates and camera rotations as well as object co-ordinates. As the engineer will be primarily concerned with the object co-ordinates and not with the camera parameters, a reasonable choice would appear to be

$$
\mathbf{P} = \begin{pmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{20}
$$

where the partitioning is compatible with that given in equation (4). Thus the identity matrix in equation (20) is of order $3n_p \times 3n_p$ and hence we only minimise the sum of the variances of the object co-ordinates. With other photogrammetric applications, a different choice of weight matrix may be appropriate. For example, with independent model triangulation it may be desirable to know the relative precision of the tie point and perspective centre co-ordinates and, in such cases, only the weights associated with the camera rotations need be set to zero.† However, problems may arise with different units of measurement if both rotation elements and co-ordinates are included in the weighted trace.

We now see how we can use these inner accuracy and free network results to compare precision. When we have no object space control, the dispersion matrix can be derived using equation (18) (or(19) if arbitrary constraints are imposed). In the case where object space control is incorporated in an attempt to improve the solution, the derived dispersion matrices can be transformed into a form where the results can be directly compared with the true free network results using equation (19).

Results

The results from some of the precision studies carried out are now presented. In all cases the precision estimates refer to the inner accuracy derived by using the seven elements of absolute orientation as the filter parameters, and referred to all the object points. Specific conclusions can only be reached for each individual practical application, but it is possible to provide very general results to act as guidelines. The written bundle program can accommodate situations involving no control, object space co-ordinates, object space distances, camera position data or camera attitude data. The accuracy of such information is conveyed through associated weight matrices. The results presented are restricted to comparing the precision when object space co-ordinates have been incorporated in the adjustment with the precision when no control is included, this being considered as one of the most useful investigations in engineering projects. However, this is not to deny that in certain close range projects the incorporation of other data may be appropriate. (Wrobel and Ellenbeck (1976) give an example of the use of observed camera orientation elements in a bundle adjustment.) In all cases a standard deviation of $\pm 3\ \mu m$ has been assumed in the picture co-ordinates, a realistic figure for engineering applications when using glass plates.

Solid cube. Consider the solid cube depicted in Fig. 6, in which each edge measures 6 m and on which 16 targets are placed on each of six faces in a regular grid pattern, giving

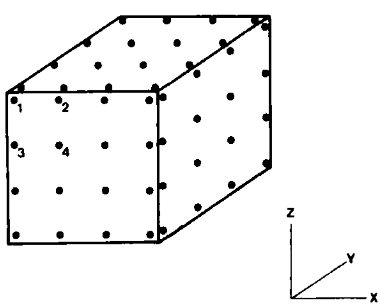

FIG. 6. The solid cube, with 16 targets on each of its six faces, used for simulation studies. Each edge of the cube measures 6 m.

a total of 96 targets in all. We now compare two different geometries. In the first case we use eight photographs in a multistation arrangement where the elongation of each camera axis passes through one corner of the cube and also through the cube's centre (compare with photographs A, B, C, D in Fig. 1, repeated on the underside of the cube; photograph C, for example, would have perspective centre co-ordinates $X_0 = 9{\cdot}0$ m, $Y_0 = 9{\cdot}0$ m,

† The author is grateful to Professor A. K. I. Torlegård for this suggestion.

$Z_0 = 9{\cdot}0$ m, based on an origin at the centre of the cube). In the second case we take one stereopair of non-convergent photographs of each face, making 12 photographs in all (compare with photographs 1 and 2 in Fig. 1, repeated on the other five sides; photograph 2, for example, would have perspective centre co-ordinates $X_0 = 3{\cdot}0$ m, $Y_0 = -13{\cdot}5$ m, $Z_0 = 0{\cdot}0$ m, again based upon the centre of the cube as origin). In both cases a principal distance of 150 mm has been assumed. With the multistation geometry we carry out three different bundle adjustments:

(i) no control points incorporated (pure relative orientation);
(ii) four corner points of each face as control points (24 in all);
(iii) all camera parameters held fixed at their true value.

With the stereopair geometry only situations (ii) and (iii) were considered, it no longer being possible to co-ordinate the whole structure in a common system without using some control points. The results of these studies for points 1, 2, 3 and 4 in Fig. 6 are presented in Table II.

TABLE II. Internal precision results for the four points numbered in Fig. 6 using different geometrical and control point configurations.

		Standard deviations (mm)					
		Eight multistation convergent photographs			*Six stereopairs (12 photographs)*		
Control information	*Point number*	*X*	*Y*	*Z*	*X*	*Y*	*Z*
No control points used	1	0·175	0·155	0·175	—	—	—
	2	0·181	0·153	0·172	—	—	—
	3	0·172	0·153	0·181	—	—	—
	4	0·179	0·152	0·179	—	—	—
24 corner points used as control	1	(0·042)	(0·042)	(0·042)	(0·089)	(0·089)	(0·089)
	2	0·180	0·151	0·169	0·198	0·612	0·252
	3	0·169	0·151	0·180	0·252	0·593	0·196
	4	0·179	0·152	0·179	0·189	0·585	0·187
All camera parameters fixed	1	0·164	0·146	0·164	0·210	0·507	0·210
	2	0·176	0·147	0·165	0·162	0·506	0·210
	3	0·165	0·147	0·176	0·210	0·506	0·162
	4	0·176	0·149	0·176	0·161	0·513	0·161

From Table II it can be seen that the multistation geometry provides fairly homogeneous precision in each direction (the homogeneity can be improved by increasing the principal distance). The inclusion of control points causes only a marginal increase in precision and, even when all the camera parameters are held fixed, the precision is only improved by a factor of 1·04 (or 4 per cent) over the case where no control points are incorporated. With the stereopair geometry, the discrepancy between the three co-ordinates is much greater, although the precision in particular co-ordinates for specific points may be no worse than in the multistation case. Note that, in order to obtain full coverage, multistation geometry may actually involve *fewer* photographs than the total required for stereopair coverage.

Poor stereopair geometry. Although it should be clear from the above example that good multistation geometry may provide for the best overall precision, in certain practical cases such designs may not be possible due to site restrictions. (However, this is not to deny the practical versatility of multistation methods in many situations.) Fig. 7 illustrates a situation encountered on a practical project where a stereopair of photographs with a poor base to distance ratio was used. The 30 targets which were used imaged on a relatively small proportion of the photographic format and it can be seen that there is a considerable depth in the object. Table III compares the inner accuracy precision for one case where no control points were used and a second case where nine control points were incorporated in the adjustment (the values corresponding to these control points are not listed). The last three columns of Table III indicate the resulting precision improvement in each co-ordinate. These values vary considerably, but the average precision improvement

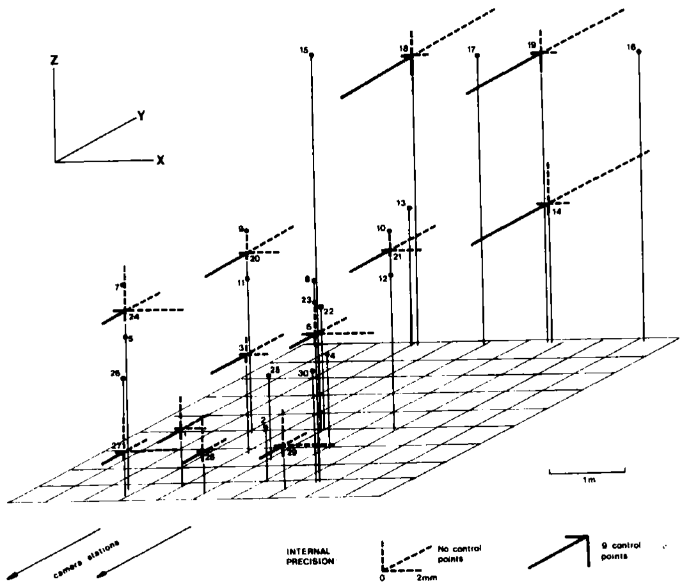

FIG. 7. The arrangement of object points when poor stereopair geometry was used. Selected internal precision results are also shown.

TABLE III. Internal precision results for the object points shown in Fig. 7.

Point number	*Standard deviation (mm)* No control points X	Y	Z	Nine control points X	Y	Z	*Precision improvement* X	Y	Z
23	3·10	2·13	2·45	0·66	1·59	0·49	4·7	1·3	5·0
24	3·09	2·09	2·34	0·61	1·57	0·48	5·1	1·3	4·9
5	3·09	1·91	1·58	0·61	1·56	0·37	5·1	1·2	4·2
6	3·09	1·90	1·46	0·65	1·57	0·36	4·8	1·2	4·1
27	2·81	1·43	1·80	0·55	1·41	0·49	5·1	1·0	3·7
28	0·37	1·46	1·96	0·24	1·40	0·51	1·6	1·0	3·8
29	2·67	1·43	1·80	0·55	1·41	0·50	4·8	1·0	3·6
1	1·36	1·70	1·65	0·33	1·59	0·47	4·1	1·1	3·5
2	1·42	1·70	1·64	0·35	1·59	0·47	4·0	1·1	3·4
25	0·26	2·20	1·10	0·22	1·95	0·39	1·2	1·1	2·8
3	1·19	2·35	0·87	0·34	2·15	0·35	3·5	1·1	2·5
4	1·20	2·35	0·89	0·34	2·15	0·36	3·5	1·1	2·5
22	0·24	2·75	0·57	0·23	2·57	0·28	1·0	1·1	2·0
20	1·77	2·86	0·90	0·51	2·58	0·32	3·5	1·1	2·8
21	1·74	2·86	0·85	0·49	2·58	0·30	3·6	1·1	2·8
11	1·75	2·75	0·35	0·51	2·58	0·24	3·4	1·1	1·4
12	1·73	2·75	0·34	0·49	2·58	0·24	3·6	1·1	1·4
18	0·95	5·00	1·19	0·60	4·54	1·05	1·6	1·1	1·1
17	0·43	4·89	1·14	0·34	4·56	1·04	1·3	1·1	1·1
19	0·94	5·00	1·19	0·54	4·55	1·05	1·8	1·1	1·1
14	1·13	6·06	2·15	0·65	4·74	0·51	1·8	1·3	4·2

factor is 3·3 in X, 1·1 in Y (the direction of the camera axes) and 3·0 in Z. Selected values have been included in Fig. 7. Thus although the improvement in the Y co-ordinate (depth) is not very great, there are dramatic improvements in the X and Z co-ordinates and, in those cases where these particular co-ordinates are of prime importance (such as a wriggle survey of a tunnel), the incorporation of control points in the bundle adjustment would be invaluable. The reason for this improvement lies in the greater precision of the camera parameters, even though the basic geometry is poor. Table IV indicates the inner accuracy precision for the camera parameters in the two cases (related to the 30 object points), and the overall precision improvement is most noticeable.

TABLE IV. Internal precision of the exterior orientation elements of the photographs used for co-ordinating the points shown in Fig. 7.

Camera station	Parameter	Standard deviation: No control points	Standard deviation: Nine control points	Precision improvement
1	X_0	3·2 mm	0·9 mm	3·4
	Y_0	10·9 mm	1·7 mm	6·6
	Z_0	6·3 mm	1·3 mm	4·9
	θ_X	584 ^{cc}	118 ^{cc}	4·9
	θ_Y	84 ^{cc}	50 ^{cc}	1·7
	θ_Z	224 ^{cc}	85 ^{cc}	2·6
2	X_0	3·3 mm	1·1 mm	3·1
	Y_0	10·9 mm	1·6 mm	6·7
	Z_0	6·3 mm	1·3 mm	4·9
	θ_X	584 ^{cc}	118 ^{cc}	4·9
	θ_Y	85 ^{cc}	51 ^{cc}	1·6
	θ_Z	224 ^{cc}	93 ^{cc}	2·4

Bundle Adjustment or Space Resection?

It has been indicated that with good multistation geometry, the incorporation of control points may be unnecessary in engineering and industrial applications of the bundle method (compare with Kenefick, 1971). However, if it happens that control points are readily available, one should, perhaps, question the necessity of adopting a bundle solution at all. An alternative method in such a case, which would incur far lower computer expense in terms of time and storage, might consist of a series of space resections followed by multiray intersections, in which the basic geometric configuration would not be changed from a multistation bundle solution. As the geometry is unchanged, the precision of the camera parameters offers the most direct means of comparison. Table V lists various results for the multistation geometry of the solid cube illustrated in Fig. 6. Of course with the same amount of control the bundle solution always provides a better result than the space resections, but the discrepancies are small enough to produce relatively minor differences when translated into object point precision. However, the bundle solution must, of course, be adopted when widespread control is not available.

TABLE V. Internal precision of the exterior orientation elements of the eight multistation photographs of the cube shown in Fig. 6 using bundle adjustment and space resection methods.

Camera parameters	Bundle adjustment: No control points	Bundle adjustment: 24 control points	Space resection: 24 control points (12 on each photograph)	Space resection: 24 control points on each photograph
X_0, Y_0, Z_0	0·305 mm	0·262 mm	0·350 mm	0·302 mm
$\theta_X, \theta_Y, \theta_Z$	13·7 ^{cc}	11·9 ^{cc}	16·0 ^{cc}	13·0 ^{cc}

COMPENSATION OF SYSTEMATIC ERRORS

The results of error propagation obtained from dispersion matrices, such as those presented in the previous section, provide an estimate of *precision*, whereas in practice we would like an estimate of *accuracy*. Leaving aside, for the moment, the problem of gross errors, the results indicated by dispersion matrices can only be used as a measure of accuracy if uncompensated systematic errors are negligible. In practice this may not be the case, because systematic errors from a variety of physical sources will be present to a greater or lesser extent if we use the basic bundle model given in equations (1). Our aim should be to reduce the magnitude of the uncompensated systematic errors to a level where they are not significant in relation to the random errors. Known discrepancies between the mathematical model (the central perspective) and the physical reality, as determined by the prior calibration of the photogrammetric system, can be compensated *a* *priori*, but residual discrepancies remain and these may attain a significant magnitude. Various methods have been developed in an attempt to compensate for these systematic errors, but the value of the *self calibration* technique in aerial triangulation has been proved in recent years (Ebner, 1976) and the method may be of equal importance in engineering photogrammetry.

Self Calibrating Bundle Adjustment

The concept of self calibration involves the extension of the mathematical model, in our case defined by equations (1), to include *additional parameters* that attempt to model the residual systematic errors inherent in the real photogrammetric system. The difference between the self calibration technique and other methods is that self calibration, as its name suggests, attempts this modelling without requiring any additional observations to be made specifically for the purpose of systematic error compensation. In close range photogrammetry, this same technique can be used to calibrate a camera, but a distinction should be made between the *self calibration of a camera*, where the main purpose is to recover the inner orientation elements and distortion parameters using little or no object space control, and a *self calibrating block adjustment*, where the aim is to reduce systematic errors in order to obtain the highest accuracies in the co-ordinates of an object. In close range photogrammetry, however, this logical distinction may become blurred, especially when non-metric cameras are employed. We will now briefly investigate the application of the self calibrating bundle adjustment to engineering and industrial photogrammetry, noting the main differences with aerial triangulation.

Some basic differences between aerial survey and engineering applications of photogrammetry have been outlined in Table I and we now see how some of these differences affect the choice of additional parameters in a self calibrating bundle adjustment. A basic choice can be made between parameters which have a physical interpretation, such as terms for residual lens distortion, and strictly empirical terms which attempt to model deformations without specifically identifying their causes. Although the tendency in aerial triangulation has been to replace physical terms with empirical ones (Brown, 1974, 1976; Ebner, 1976; Heikkilä and Inkilä, 1978; Schut, 1979), a similar selection may not be valid for engineering applications. We may outline some of the reasons.

Film and plates. Many of the additional parameters used in aerial triangulation may be compensating for film deformations. In high accuracy engineering applications glass plates should be used whenever possible and thus the value of empirical polynomial terms becomes more questionable.

Standardised photography. Assumptions concerning the photography, which are justified in aerial triangulation, may be inappropriate in engineering applications. For example, the orthogonal bivariate polynomials used by Ebner (1976) assume; (i) vertical photography; (ii) flat terrain; and (iii) a regular image point distribution. All three assumptions may be false in engineering photogrammetry, although Grün (1978b) proposes the adoption of such parameters in close range applications.

Interior orientation elements. In aerial triangulation the inner orientation elements (or parameters associated with them) cannot be recovered with any degree of certainty in the block adjustment. This is because, when nominally vertical photographs are taken of essentially flat terrain, these elements are highly correlated with the exterior orientation parameters. Thus the inner orientation elements have to be determined by a camera calibration which is separate from the adjustment process. However, any residual errors that exist in the principal distance or principal point co-ordinates will have little effect because the process of *projective compensation* means that the exterior orientation parameters will be correspondingly amended, so that the final accuracy is not significantly diminished. On the other hand, projective compensation does not apply to nearly the same extent when using multistation convergent photography of a spatial object and, in such cases, inner orientation errors may cause the accuracy of the final solution to be significantly worse than that indicated by dispersion matrices (Konecny, 1965). However, a decrease in the effect of projective compensation implies an increase in the ability to determine the inner orientation elements. Thus convergent multistation photography may permit the recovery of the inner orientation elements, even if there are no control points, because of the much lower correlations between the interior and exterior orientation elements. Indeed, this is the basis of the self calibration methods for calibrating close range cameras (Kenefick *et al.*, 1972; Kölbl, 1972; Bhatti, 1973; Faig, 1975) and a self calibrating bundle adjustment program which incorporates inner orientation and distortion parameters in its additional parameter set can be used for camera calibration purposes.

The selection of a suitable additional parameter set for engineering applications of the bundle adjustment requires further investigation, although the interior orientation elements should be included initially. Once selected, the additional parameters can be added to equations (1). The partitioned normal equations given by equation (4) will then be augmented to the form

$$\begin{pmatrix} \mathbf{N}_p & \mathbf{N}_{ps}^{\mathrm{T}} & \mathbf{N}_{pa}^{\mathrm{T}} \\ \mathbf{N}_{ps} & \mathbf{N}_s & \mathbf{N}_{sa}^{\mathrm{T}} \\ \mathbf{N}_{pa} & \mathbf{N}_{sa} & \mathbf{N}_a \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x}_p \\ \Delta\mathbf{x}_s \\ \Delta\mathbf{x}_a \end{pmatrix} = \begin{pmatrix} \mathbf{t}_p \\ \mathbf{t}_s \\ \mathbf{t}_a \end{pmatrix}, \tag{21}$$

where the suffix a denotes the additional parameters. If there are n_a additional parameters, the coefficient matrix of the reduced normal equations will be of order $(6n_s+n_a)\times(6n_s+n_a)$.

In common with aerial triangulation, however, we must guard against taking too many parameters, especially when including interior orientation elements. This can be accomplished by the following means.

(i) The additional parameters should be treated as observations with *a priori* weights. This is easily done when using the unified method of least squares.

(ii) The value of each additional parameter (including inner orientation elements) should be tested statistically and the parameter should be eliminated if not significant.

The statistical testing of additional parameters requires the computation of certain elements of the dispersion matrix. However, such testing is made more difficult when high correlations exist either between the additional parameters themselves, or between the additional parameters and the camera parameters or object co-ordinates. The advantage of orthogonal bivariate polynomials in aerial triangulation lies in their relatively low correlations but, as we have already noted, such sets may be inappropriate for engineering projects. On the other hand, the inclusion of interior orientation elements into an *initial* adjustment is to be recommended, although statistical testing of these elements must take into account their correlation (to a greater or lesser degree, depending upon the geometry) with the exterior orientation elements.

If an additional parameter has a very high correlation with other parameters ($|\rho|>0{\cdot}97$, say, ρ being the coefficient of correlation), it should be rejected from the adjustment. If several additional parameters have high (but not very high) correlations ($|\rho|>0{\cdot}75$, say), they should be tested together using multidimensional methods. Suitable statistical procedures have been outlined by Grün (1978a, b, c).

The importance of statistical testing of the additional parameters cannot be overemphasised, especially when including inner orientation elements in engineering applications. When these parameters are significant, their inclusion may provide for very considerable improvements in accuracy, but their inclusion in situations where they cannot be determined to any great precision can be expected to lead to worse and not better results. Unless it is known by sound knowledge of the geometrical configuration that the inner orientation elements can be determined (and therefore should be included), or cannot be determined (and therefore should be eliminated), this choice should be left to objective statistical procedures.

Detection of Gross Errors

Although the elimination of gross errors is an important aspect of practical photogrammetry, it is not intended to discuss this topic in detail other than to point out the advantages of multistation bundle configurations in detecting such errors.

Before data are introduced into a bundle adjustment program, they should be checked for gross errors as far as is practicable. However, this checking can never guarantee the elimination of all gross errors (particularly small ones) prior to the bundle adjustment, and we therefore ask ourselves two questions concerning the *reliability* of the system.

(i) Which remaining gross errors can be detected during the adjustment? (In other words, what is the *internal reliability*?)

(ii) What are the effects of undetected gross errors upon the accuracy of the object co-ordinates? (In other words, what is the *external reliability*?)

Methods of gross error *detection* should be concerned with improving the internal reliability; external reliability is concerned with effect, not detection.

A popular method of checking for gross errors after an initial adjustment is to compare the weighted residuals with $\hat{\sigma}_0^2$, the reference variance.†

$$\hat{\sigma}_0^2 = (\mathbf{v}^{\mathrm{T}}\mathbf{W}\mathbf{v})/r \tag{22}$$

where $\mathbf{v}$ is the vector of residuals, $\mathbf{W}$ is the weight matrix of the observations and r is the redundancy in the system. For example, if

$$|v_i w_{ii}^{\frac{1}{2}}| > 3\hat{\sigma}_0 \tag{23}$$

we suspect a gross error in the ith observation. However, such a technique assumes that all the residuals can be represented by a common *a posteriori* variance. The fallacy of such an assumption is easily demonstrated by a situation which occurs frequently in conventional stereophotogrammetry. When an object point (which is not a control point) is intersected by rays from only two photographs, an error (be it random, systematic or gross) along the corresponding epipolar lines of the two photographs cannot be detected. This is why, of course, we are so preoccupied with y parallax in the relative orientation of two non-convergent photographs; the epipolar lines are aligned (approximately) parallel to the x co-ordinate axis of the picture and consequently any small error in the x picture co-ordinate cannot be detected.

Grün (1978a, b, 1979) has suggested the adoption of the *data snooping* technique for gross error detection in photogrammetry. The method is based upon $\mathbf{Q}_{vv}$, the *a posteriori* dispersion matrix of the observations.

$$\mathbf{Q}_{vv} = \mathbf{W}^{-1} - \mathbf{A}\mathbf{N}^{-1}\mathbf{A}^{\mathrm{T}}. \tag{24}$$

$\mathbf{A}$ is the coefficient (design) matrix of the observation equations and $\mathbf{N}$ is the coefficient matrix of the normal equations. The diagonal elements of $\mathbf{Q}_{vv}$, q_{vv}, can be used in a Student t-test of the individual residuals v_i using the test statistic

$$u_i = \frac{v_i}{\hat{\sigma}_{v_i}}, \tag{25}$$

where

$$\hat{\sigma}_{v_i}^2 = \hat{\sigma}_0^2\, q_{v_i v_i}. \tag{26}$$

Such techniques require a considerable amount of computation, although the problem is not so severe in engineering applications of photogrammetry as it is with the large blocks of photography encountered in aerial triangulation. (Suggestions for making the method more practicable have been outlined by Grün (1979).) In contrast to the common criterion (23), the design matrix $\mathbf{A}$ is taken into account in (24) and thus provides a much more satisfactory criterion for the detection of gross errors.

Gross Error Detection with Multistation Geometry

The test statistic u_i given in equation (25) indicates the lack of reliability encountered in the x co-ordinates of conventional stereophotogrammetry, for in this case $v_i \simeq 0$ and $q_{v_i v_i} \simeq 0$, so that u_i is indeterminant. Thus the magnitude of a gross error that may go

† "Reference variance" is the term used by Mikhail (1976). A more common expression is the "variance of an observation of unit weight", but this has unfortunate implications. Thompson (1968) suggests alternative names.

undetected in such cases may have serious consequences for the accuracy of the final object co-ordinates (that is, the external reliability will be relatively poor). On the other hand, multistation methods provide much better internal reliability, because each image is related to more than one epipolar line and thus any errors are much more likely to be detected (Fig. 8). (Further examples can be found in Grün (1978b).) Thus any gross errors that will go undetected with multistation bundle methods will be very small, with minimal consequences for object point accuracy.

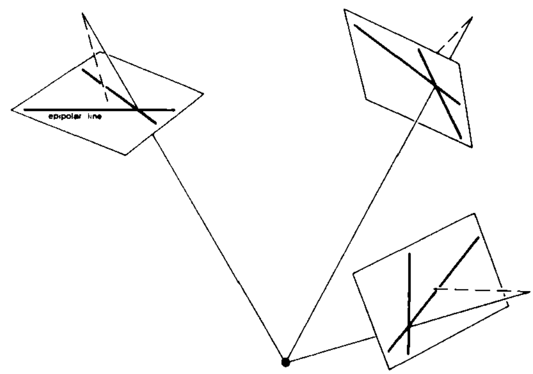

FIG. 8. With multistation geometry each image point is related to more than one epipolar line.

With fairly regular multistation configurations the necessity of using criteria such as (24) to (26) is reduced, because the values of $q_{v_i v_i}$ are of similar magnitude in such cases. For example, the values of $\hat{\sigma}_0(q_{vv})^{\frac{1}{2}}$ for the multistation photography of the cube shown in Fig. 6 ranged from 1·82 μm to 2·50 μm (the *a priori* standard deviation being 3·00 μm). In such cases, reversion to non-rigorous criteria such as those given in (23) may be justified in practice, although the data snooping technique remains preferable from a theoretical standpoint.

CONCLUSIONS

The bundle adjustment method is a powerful computational technique which provides the necessary flexibility which is essential in the various situations that may be encountered when co-ordinating engineering and industrial structures by photogrammetric methods. This paper has demonstrated that the application of the bundle method to multistation photography can provide high, homogeneous precision, even without the incorporation of control data. Such photography may be more sensitive to errors in the inner orientation elements, but it *may* be possible to recover these elements using the self calibration technique. However, multistation methods are essential if the photogrammetric system is to be made more reliable by detecting small gross errors. In geometrically weak situations, which may have to be tolerated due to site restrictions in certain instances, the incorporation of control information into the bundle adjustment can be invaluable in improving the precision, although lower accuracies are inherent with poor geometry.

Much of the content of this paper has deliberately been of a theoretical nature. The concepts of inner accuracy and free network adjustments, which have received widespread attention in geodesy in recent years, can also be of value in precision estimation in analytical photogrammetry. However, considerations of systematic and gross errors will require extensive practical testing and verification before firm conclusions can be reached.

Photogrammetry is concerned with "obtaining reliable measurements by means of photography".† Above all, this paper has demonstrated that the methods of obtaining

† Thompson, M. M. (Ed)., 1966. *Manual of photogrammetry*. Third edition. American Society of Photogrammetry. See page 1.

reliable measurements from photography of an engineering structure need bear little resemblance to the most widespread application of photogrammetry, that of mapping from aerial photography.

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APPENDIX A

The linearisation of equations (1) (also considering equations (2) and (3)) using Newton's method results in the following pair of equations for each surface image:

$$\begin{aligned} & a_x \Delta X_i + b_x \Delta Y_i + c_x \Delta Z_i - a_x \Delta X_0 - b_x \Delta Y_0 - c_x \Delta Z_0 \\ & \quad + d_x \theta_X + e_x \theta_Y + g_x \theta_Z = x_i - k_x, \\ & a_y \Delta X_i + b_y \Delta Y_i + c_y \Delta Z_i - a_y \Delta X_0 - \Delta b_y \Delta Y_0 - c_y \Delta Z_0 \\ & \quad + d_y \theta_X + e_y \theta_Y + g_y \theta_Z = y_i - k_y, \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} a_x &= x_i r'_{13} - f r'_{11}, & a_y &= y_i r'_{13} - f r'_{12}, \\ b_x &= x_i r'_{23} - f r'_{21}, & b_y &= y_i r'_{23} - f r'_{22}, \\ c_x &= x_i r'_{33} - f r'_{31}, & c_y &= y_i r'_{33} - f r'_{32}, \\ d_x &= b_x(Z_i - Z_0) - c_x(Y_i - Y_0), & d_y &= b_y(Z_i - Z_0) - c_y(Y_i - Y_0), \\ e_x &= c_x(X_i - X_0) - a_x(Z_i - Z_0), & e_y &= c_y(X_i - X_0) - a_y(Z_i - Z_0), \\ g_x &= a_x(Y_i - Y_0) - b_x(X_i - X_0), & g_y &= a_y(Y_i - Y_0) - b_y(X_i - X_0), \end{aligned} \tag{A.2}$$

and r'_{ij} is a typical element of the orthogonal matrix $\mathbf{R}'$ in equation (2); $\Delta X_i, \Delta Y_i, \Delta Z_i$ are the corrections to the approximate space co-ordinates of the object points; $\Delta X_0, \Delta Y_0, \Delta Z_0$ are the corrections to the approximate space co-ordinates of the perspective centre; k_x and k_y are the numerical values of the right hand sides of equations (1) determined at the approximation values.

Let the total coefficient matrix of the linearised observation equations be represented by **A**. The two rows of **A** associated with equations (A.1) can be represented by two vectors $\mathbf{h}_x^T$ and $\mathbf{h}_y^T$ where

$$\begin{aligned} \mathbf{h}_x^T &= (0, \dots, 0, a_x, b_x, c_x, 0, \dots, 0, \\ & \qquad -a_x, -b_x, -c_x, d_x, e_x, g_x, 0, \dots, 0), \\ \mathbf{h}_y^T &= (0, \dots, 0, a_y, b_y, c_y, 0, \dots, 0, \\ & \qquad -a_y, -b_y, -c_y, d_y, e_y, g_y, 0, \dots, 0). \end{aligned} \tag{A.3}$$

APPENDIX B

Let the system of linearised observation equations be given by

$$\mathbf{v} = \mathbf{A}\Delta\mathbf{x} - \mathbf{l}, \tag{B.1}$$

where $\mathbf{v}$ is a vector of residuals to the observations contained in $\mathbf{l}$ and $\mathbf{A}$ is a coefficient matrix. Let us divide the solution vector $\Delta\mathbf{x}$ into two parts such that

$$\Delta\mathbf{x} = \Delta\mathbf{x}'' + \mathbf{G}\Delta\mathbf{t}, \tag{B.2}$$

where $\Delta\mathbf{t}$ is a vector of *filter parameters* and $\mathbf{G}$ is a corresponding *filter matrix*. In a free network adjustment, $\Delta\mathbf{t}$ will correspond to some or all of the translational, rotational and scaling elements of the (arbitrary) co-ordinate system. We look for a vector $\Delta\mathbf{x}''$ from which the effect of the filter parameters has been eliminated, but this vector must not alter the vector or residuals in equation (B.1). Thus

$$\mathbf{v} = \mathbf{A}\Delta\mathbf{x}'' - \mathbf{l} \tag{B.3}$$

must always be satisfied. Substituting (B.2) in (B.1) we find that (B.3) is only satisfied if

$$\mathbf{AG}\Delta\mathbf{t} = \mathbf{0}. \tag{B.4}$$

As $\Delta\mathbf{t}$ is arbitrary, (B.4) implies

$$\mathbf{AG} = \mathbf{0}. \tag{B.5}$$

(See also Meissl (1969) and Mittermayer (1972b).)

Having selected $\mathbf{G}$ in a suitable manner (by satisfying equation (B.5)), we look for a particular solution vector $\Delta\mathbf{x}'$, say, which has the property of being the minimum weighted Euclidean norm solution

$$\Delta\mathbf{x}'^{\mathrm{T}}\,\mathbf{P}\Delta\mathbf{x}' \to \min \tag{B.6}$$

or of producing the minimum weighted trace of $\mathbf{Q}$, the dispersion matrix of $\Delta\mathbf{x}'$

$$\mathrm{tr}\,(\mathbf{QP}) \to \min \tag{B.7}$$

($\mathbf{P}$ is an arbitrary weight matrix). It is easily shown that equations (B.6) and (B.7) are equivalent (see Ebner (1974), for example). For our particular solution $\Delta\mathbf{x}'$, equation (B.2) reads

$$\Delta\mathbf{x} = \Delta\mathbf{x}' + \mathbf{G}\Delta\mathbf{t}. \tag{B.8}$$

If we choose to treat equation (B.8) as an observation equation with residuals $\Delta\mathbf{x}'$, condition (B.6) (and thus (B.7)) will correspond to the well known least squares solution

$$\Delta\mathbf{t} = (\mathbf{G}^{\mathrm{T}}\mathbf{PG})^{-1}\,\mathbf{G}^{\mathrm{T}}\,\mathbf{P}\Delta\mathbf{x}. \tag{B.9}$$

We now distinguish between two cases. In the first case the coefficient matrix of the normal equations derived from equation (B.1) is singular. Substituting equation (B.8) in (B.9) (for $\Delta\mathbf{x}$) leads to the condition equation

$$(\mathbf{G}^{\mathrm{T}}\mathbf{PG})^{-1}\,\mathbf{G}^{\mathrm{T}}\,\mathbf{P}\Delta\mathbf{x}' = \mathbf{0}. \tag{B.10}$$

As $\mathbf{G}^{\mathrm{T}}\,\mathbf{PG}$ is regular, equation (B.10) implies

$$\mathbf{G}^{\mathrm{T}}\,\mathbf{P}\Delta\mathbf{x}' = \mathbf{0}. \tag{B.11}$$

The least squares solution of the observation equations (B.1) together with the condition equations (B.11) leads to the well known system of normal equations

$$\begin{pmatrix} \mathbf{A}^{\mathrm{T}}\,\mathbf{WA} & \mathbf{PG} \\ \mathbf{G}^{\mathrm{T}}\,\mathbf{P} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{x}' \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^{\mathrm{T}}\,\mathbf{Wl} \\ \mathbf{0} \end{pmatrix}, \tag{B.12}$$

where $\mathbf{W}$ is the weight matrix of the observations and $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers. The coefficient matrix of equation (B.12) is non-singular and can therefore be inverted to derive the dispersion matrix $\mathbf{Q}$ corresponding to $\Delta\mathbf{x}'$. (See also Mittermayer (1972b) and Meissl (1969).)

In the second case, the dispersion matrix of $\Delta\mathbf{x}$, $\mathbf{C}$, say, is already available (perhaps due to the use of arbitrary constraints). Substituting equation (B.9) in (B.8) (for $\Delta\mathbf{t}$) leads to the equation

$$\Delta\mathbf{x}' = (\mathbf{I} - \mathbf{G}(\mathbf{G}^{\mathrm{T}}\,\mathbf{PG})^{-1}\,\mathbf{G}^{\mathrm{T}}\,\mathbf{P})\,\Delta\mathbf{x}. \tag{B.13}$$

Applying the law of propagation of errors to equation (B.13), the required dispersion matrix is given by

$$\mathbf{Q} = (\mathbf{I} - \mathbf{G}(\mathbf{G}^{\mathrm{T}}\,\mathbf{PG})^{-1}\,\mathbf{G}^{\mathrm{T}}\,\mathbf{P})\mathbf{C}(\mathbf{I} - \mathbf{PG}(\mathbf{G}^{\mathrm{T}}\,\mathbf{PG})^{-1}\,\mathbf{G}^{\mathrm{T}}). \tag{B.14}$$

(See also Meissl (1964, 1969) and Ebner (1974).)

APPENDIX C

Let the matrix **G** in equation (15) be partitioned into seven vectors

$$\mathbf{G} = (\mathbf{g}_{tX}\,\mathbf{g}_{tY}\,\mathbf{g}_{tZ}\,\mathbf{g}_{rX}\,\mathbf{g}_{rY}\,\mathbf{g}_{rZ}\,\mathbf{g}_s).$$

The elements of these vectors are given (in row form) in equations (16) and (17). Here $\mathbf{g}_{tX}$, $\mathbf{g}_{tY}$ and $\mathbf{g}_{tZ}$ represent translations in the X, Y and Z directions respectively, $\mathbf{g}_{rX}$, $\mathbf{g}_{rY}$ and $\mathbf{g}_{rZ}$ represent rotations about the X, Y and Z axes respectively, and $\mathbf{g}_s$ represents the scale. From equations (15) to (17) and (A.1) to (A.3) we then have the following equations.

Translations

$$\mathbf{h}_x^{\mathrm{T}}\mathbf{g}_{tX} = a_x - a_x = 0, \qquad \mathbf{h}_y^{\mathrm{T}}\mathbf{g}_{tX} = a_y - a_y = 0,$$
$$\mathbf{h}_x^{\mathrm{T}}\mathbf{g}_{tY} = b_x - b_x = 0, \qquad \mathbf{h}_y^{\mathrm{T}}\mathbf{g}_{tY} = b_y - b_y = 0,$$
$$\mathbf{h}_x^{\mathrm{T}}\mathbf{g}_{tZ} = c_x - c_x = 0, \qquad \mathbf{h}_y^{\mathrm{T}}\mathbf{g}_{tZ} = c_y - c_y = 0,$$

Rotations

$$\mathbf{h}_x^{\mathrm{T}}\mathbf{g}_{rX} = (-b_x Z_i + c_x Y_i) + (b_x Z_0 - c_x Y_0 + b_x(Z_i - Z_0) - c_x(Y_i - Y_0)) = 0,$$
$$\mathbf{h}_x^{\mathrm{T}}\mathbf{g}_{rY} = (a_x Z_i - c_x X_i) + (-a_x Z_0 + c_x X_0 + c_x(X_i - X_0) - a_x(Z_i - Z_0)) = 0,$$
$$\mathbf{h}_x^{\mathrm{T}}\mathbf{g}_{rZ} = (-a_x Y_i + b_x X_i) + (a_x Y_0 - b_x X_0 + a_x(Y_i - Y_0) - b_x(X_i - X_0)) = 0,$$
$$\mathbf{h}_y^{\mathrm{T}}\mathbf{g}_{rX} = (-b_y Z_i + c_y Y_i) + (b_y Z_0 - c_y Y_0 + b_y(Z_i - Z_0) - c_y(Y_i - Y_0)) = 0,$$
$$\mathbf{h}_y^{\mathrm{T}}\mathbf{g}_{rY} = (a_y Z_i - c_y X_i) + (-a_y Z_0 + c_y X_0 + c_y(X_i - X_0) - a_y(Z_i - Z_0)) = 0,$$
$$\mathbf{h}_y^{\mathrm{T}}\mathbf{g}_{rZ} = (-a_y Y_i + b_y X_i) + (a_y Y_0 - b_y X_0 + a_y(Y_i - Y_0) - b_y(X_i - X_0)) = 0.$$

Scale

$$\begin{aligned}
\mathbf{h}_x^{\mathrm{T}}\mathbf{g}_s &= a_x X_i + b_x Y_i + c_x Z_i - a_x X_0 - b_x Y_0 - c_x Z_0 \\
&= (x_i r'_{13} - f r'_{11})(X_i - X_0) + (x_i r'_{23} - f r'_{21})(Y_i - Y_0) + (x_i r'_{33} - f r'_{31})(Z_i - Z_0) \\
&= x_i(r'_{13}(X_i - X_0) + r'_{23}(Y_i - Y_0) + r'_{33}(Z_i - Z_0)) \\
&\qquad - f(r'_{11}(X_i - X_0) + r'_{21}(Y_i - Y_0) + r'_{31}(Z_i - Z_0)) \\
&= 0 \quad \text{(using equation (1a) when } \Delta\mathbf{R} = \mathbf{I} \text{ in equation (2)).}
\end{aligned}$$

$$\begin{aligned}
\mathbf{h}_y^{\mathrm{T}}\mathbf{g}_s &= a_y X_i + b_y Y_i + c_y Z_i - a_y X_0 - b_y Y_0 - c_y Z_0 \\
&= (y_i r'_{13} - f r'_{12})(X_i - X_0) + (y_i r'_{23} - f r'_{22})(Y_i - Y_0) \\
&\qquad + (y_i r'_{33} - f r'_{32})(Z_i - Z_0) \\
&= y_i(r'_{13}(X_i - X_0) + r'_{23}(Y_i - Y_0) + r'_{33}(Z_i - Z_0)) \\
&\qquad - f(r'_{12}(X_i - X_0) + r'_{22}(Y_i - Y_0) + r'_{32}(Z_i - Z_0)) \\
&= 0 \quad \text{(using equation (1b) when } \Delta\mathbf{R} = \mathbf{I} \text{ in equation (2)).}
\end{aligned}$$

Collectively, these equations show that equation (14) is satisfied.

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Résumé

La compensation par faisceaux est un outil fort intéressant pour les applications de la photogrammétrie à l'industrie et à l'ingénerie. On recherche ici, en remplaçant surtout d'un point de vue théorique, quelles doivent être alors, s'il en *est besoin, les caractéristiques du canevas d'appui. On montre que les concepts d'exacitude interne et de "libre compensation" (équation d'observation indeterminée) présentent un intérêt certain pour ce genre d'étude. On tente également d'évaluer l'intérêt de l'étalouvage et la chambre de prise de vue, et on discute le problème de la détection des fautes.*

Zusammenfassung

Die Bündelmethode ist ein sehr flexibles analytisches Verfahren zur Koordinatenbestimmung beim Ingenieurbau und bei industriellen Konstruktionen. In der Arbeit wird, hauptsächlich vom theoretischen Gesichtspunkt, untersucht, wieviel Objektpasspunkte in einer Bündelausgleichung benötigt werden. Der Autor plädiert für die Anwendung der Multistandpunktaufnahme bei solchen Projekten. Innere Genauigkeit und freie Netzausgleichung werden als wertvolle Begriffe bei vergleichenden Genauigkeitsanalysen angesehen. Die Methoden der Selbstkalibrierung zur Eliminierung systematischer Fehler werden ebenso wie die Verfahren zur Erkennung grober Fehler diskutiert.

DISCUSSION

Mr. Proctor (*Chairman*): I have, in this very room on more than one occasion, advocated the analytical approach to various photogrammetric tasks because of that flexibility which Mr. Granshaw has demonstrated tonight. Even in topographic applications, you are not restricted to particular tilts or particular focal lengths. There are times when only an analytical solution will work. I think tonight's lecture has been a revelation in terms of a more versatile approach. You should no longer consider how many stereopairs are needed, each pair being a conventional pair of photographs with the axes approximately parallel, but should take a much wider approach; you can have camera stations all over the place, with axes in different directions, provided only that you can position the camera and take the photographs.

For some tasks you can design very strongly. You can say where the cameras are to be, which camera to use and what angular field of view is needed for a strong solution. Some of the flexibility that Mr. Granshaw has spoken of, however, comes when you have no choice of camera stations. An example is where the photographs concerned were taken years before and the object being measured is no longer in existence; you then have to use the photographs that happen to be available to you. It is thus necessary to take an approach rather like the one we have heard about tonight.

Mr. Crompton: When you make the observations on the photographs, do you view them stereoscopically? Is it possible, in general, to make stereoscopic measurements? If not, is there now a future for half a stereocomparator at half the price?

Mr. Granshaw: Perhaps I did not emphasise this point enough. If at all possible, targets should be used in such applications. If you are after very high accuracy, I do not think there can be any substitute for targetted points. This is not a commonly adopted situation. Kenefick (1977), when co-ordinating parts of a ship, normally attaches targets to the points. In such a case, of course, you can measure on a monocomparator rather than a stereocomparator. It may be possible to use stereoscopy but it is likely that you will encounter difficulties when you have the highly convergent photographs which are necessary for geometrical strength. It is something of a paradox that stereoscopy, using one's eyes, is concerned more with qualitative analysis of depths rather than precise quantitative analysis.

Mr. Burnside: You cannot use stereoscopy for the identification of points on different photographs, can you? I think you are confined to using premarked points and therefore it is a stereometric system, but stereoscopy cannot be used for the transfer of points accurately.

Mr. Granshaw: That is correct.

Mr. Newton: Can Mr. Granshaw tell us how close we are to the situation where we shall be able to go out with an ordinary amateur camera, photograph an object from random camera positions and obtain accurate measurements?

Mr. Granshaw: In this case you have to use a self calibrating bundle adjustment as the basic model. I think we are a long way from achieving the goal of obtaining high accuracy from an amateur camera.

Mr. Newton: Would medium accuracy be possible?

Mr. Granshaw: Even for medium accuracy you will have to use a very sophisticated additional parameter set in the bundle adjustment and include these parameters for each photograph rather than using a common set for all photographs. Computationally, I do not know whether the method is particularly feasible, but it could certainly be developed.

Mr. Cooper: You mentioned that it would be desirable in some cases to incorporate information about the control in the bundle adjustment. Does this information include the variance–covariance matrix of the control co-ordinates as well as the estimates of the co-ordinates from whatever method you have used to measure the control.

Mr. Granshaw: Yes, it does, but not the entire variance–covariance matrix of the measured co-ordinates. In the partitioned coefficient matrix of the normal equations, $\mathbf{N}_p$ represents the terms associated with the object points. In $\mathbf{N}_p$ the non-zero elements are the 3×3 submatrices and in $\mathbf{N}_s$ the non-zero elements are the 6×6 submatrices for the camera parameters. In this method you can include the variance–covariance matrix for control points but only if you use a 3×3 variance–covariance submatrix for each control point.

Mr. Cooper: So you do not allow for covariances between co-ordinates of different control points.

Mr. Granshaw: You can do this, but it would mean non-zero terms off the 3×3 submatrices along the diagonal of $\mathbf{N}_p$ and would upset the solution by the reduced matrix $\mathbf{N}_r$ because the inversion of $\mathbf{N}_p$ would then require something more than the inversion of a series of 3×3 matrices.

Mr. Cooper: Covariances between co-ordinates of different points are often smaller.

Mr. Granshaw: Yes, that is a practical limitation.

Mr. Cooper: As you correctly said, the bundle adjustment method can have a high intrinsic strength. You can use that to improve the precision of the control co-ordinates by incorporating them as measurements in the adjustment.

Mr. Granshaw: Yes, I think the point Kenefick is making is that the photogrammetry should not be contaminated by the surveying. The photogrammetry is effectively much better than the survey, so forget about the survey and just use the photogrammetry. If you have strong multistation geometry, then this is a valid way of going about things, but if the geometric strength is not there, you should include the surveyed points as control.

Mr. Mason: Following on from that question, have you attempted a simultaneous adjustment of the photogrammetry and the control?

Mr. Granshaw: No, I have not done this, but I believe Brown (1974) has discussed simultaneous geodetic and photogrammetric adjustments for aerial triangulation. If you are talking about aerial triangulation, then the quality of the ground survey is usually very good in comparison with the photogrammetry. It is when you come to engineering structures and similar objects that the quality of the survey is in question.

Dr. Allan: Did you look at the possible locations of control in the cases with bad geometry?

Mr. Granshaw: Yes, if you are photographing an object, then you really want the control to be at the extremities of the object. It may be that for certain engineering structures some parts may be determined very strongly by the photogrammetry but other parts may be rather weak. In such cases control, or just distances between targets, can significantly help in improving the strength of the determination.

Dr. Allan: So, in the example you gave, you chose the points to be in the best places.

Mr. Granshaw: Yes, they were fairly well distributed. There is no purpose in having control points very close together.

Chairman: Not having done the analytical study that you have, it seems intuitive to me that in the bad geometry cases, which normally mean that the camera stations are close together and the cones are narrow, the critical factor that is often missing is that there should be depth in the control. The separation of the control should be along the camera axis. Sometimes, of course, what you are doing resembles architectural photogrammetry without properly separated camera stations. The object may be the flat face of a building, but of course you are not always interested in the depth co-ordinate. This seems to be the situation which is critical and when you would perhaps need to know the interior orientation elements of the camera in advance, rather than solving for these in the adjustment.

Mr. Granshaw: Yes. I think I should emphasise the point that you cannot recover the interior orientation elements in all cases. Highly convergent photography of an object with a considerable depth range is required to recover the interior orientation elements with any accuracy. This is why some sort of statistical analysis is required. You cannot include these elements in every case, because the accuracies would significantly decrease.

Dr. Allan: Have you just used normal angle lenses, or have you examined other possibilities?

Mr. Granshaw: I have used normal and wide angle cameras. From the point of view of multistation photography, there are certainly advantages in using a normal angle camera; you find the results are then more homogeneous if you have convergent photographs. This is beneficial if homogeneity is the purpose of the exercise; the smaller the angle of the cone the better. However, in a practical situation, you may only have a certain camera available or you may not be able to get far enough away from the object to utilise a normal angle camera.

Major Neale: Have you considered, in the control, the co-ordination of the camera stations as against points in the object space?

Mr. Granshaw: Yes. With very good geometry, I gave an example where the camera parameters were held fixed which approximates the case where you can observe these parameters accurately) which is possible in certain situations but not with all engineering structures. In such situations there is not a significant improvement over the case where you do not have any control. However, in geometrically poor situations, if you compare one case incorporating control points with another involving a knowledge of the camera parameters, then if you have a fairly good distribution of control points the precision is not significantly improved by knowing the camera parameter data. They give the same order of precision in the final results.

Chairman: In your knowledge of the camera parameters, are you assuming both the orientation of the camera, which is not always easy to determine, and the co-ordinates of the perspective centre, which in close range work means the external node of the lens? This is not critical in the very distant case of aerial photography where 0·5 m in the flying height is of little consequence.

Mr. Granshaw: Work has been done on this at Hannover I believe (Wrobel and Ellenbeck, 1976) but it is not such a practical proposition for many engineering applications.

Mr. Cooper: You mentioned the practical problem of inverting the matrix of normal equations, taking the usual case where it is not rank deficient and you are not finding the pseudoinverse. This is not particularly large for the type of photogrammetry which we have been discussing. Have you looked into the possibility of doing sequential adjustments using a recursion formula for updating the inverse of the matrix. This can be useful when you are looking at the effects of different control and other measurements on your solution. You take a few readings and invert the matrix. Then you take a few more readings, update the inverse and see what effect the new readings have had on the solution and its precision.

Mr. Granshaw: No, I have not done that. Perhaps that is something to look at in the future.

Dr. Allan: On a similar subject, I know you have looked a little at the generalised inverse and its other equivalents. Would you care to say something about this, because I have come to it through Bjerhammar's work? Although it looks fine on the printed page (if you can understand it), the computational effort is massive and others, such as Meissl, have shown that this is all unnecessary.

Mr. Granshaw: Yes, I think that the generalised inverse as described by Bjerhammar and others is fine mathematically, but as a practical computational technique it is virtually dead. With large, or even medium size matrices you have to perform so many operations that it is just not practicable. The condition of the equations decreases quite considerably. One of the methods I mentioned was to augment the normal equations. If you have a rank defect of seven, as you will do in relative orientation, then you just include seven more rows and columns in the matrix of normal equations. The matrix is then non-singular and can be inverted. That is quite a practical method and the one preferred. There is a difference between deriving the equations and using them. The generalised inverse is only feasible if you have a very small system of equations.

Mr. Crompton: In your diagram (Fig. 3) of the pattern of non-zero elements in the partitioned matrix of normal equation coefficients, was the pattern derived for a given case?

Mr. Granshaw: This is not an example from a real case, but it is fairly representative. If you pack the matrix, you do not have to store all the zero submatrices in $\mathbf{N}_{ps}$. The storage requirements are all in the $\mathbf{N}_{ps}$ matrix. The more photographs you have, the more likely it is that a given point will not image on all of the photographs.

Mr. Crompton: So, if all points imaged on all photographs, $\mathbf{N}_{ps}$ would be full?

Mr. Granshaw: That is correct. Often it is an irregular pattern, unlike aerial triangulation. When you reduce them, there are still a few zero submatrices in $\mathbf{N}_r$ but not many.

Chairman: We have had a most interesting evening. We have seen from the last few questions that we are trying to stretch our speaker into areas which he has yet to cover. He may find this of value for future investigations and hopefully it may give him some ideas but this is hardly the occasion to pursue such a course. I would like to thank you, Mr. Granshaw, for a stimulating evening. To discuss a subject like this and manage to create the parallel between what you are doing and aerial triangulation without stumbling over the words "aerial triangulation" and using them to describe something which is quite patently not aerial triangulation is quite an achievement. I have no doubt we shall be hearing from you again in the future. Thank you very much indeed.