Cross Ratios

Contents

1 Projective Geometry and Cross ratios

Definition 1. The *projective plane* $\mathbb{P}^2$ is the set of lines through an observation point O in three dimensional space. A *projective line* l is a plane passing through O, and a *projective point* P is a line passing through O. If the line defining P is contained in the plane defining l, we say that $P \in l$.

If $\mathbb{A}^2$ is an ordinary plane which does not pass through O, then we can identify most projective points of $\mathbb{P}^2$ with ordinary points on $\mathbb{A}^2$ by taking the intersection of the line defining the projective point with $\mathbb{A}^2$. The projective line which is defined by a plane passing through O and parallel to $\mathbb{A}^2$ is called the *line at infinity*, or the *horizon line*. Projective points contained in the line at infinity are called *infinite points*.

If we take $O = (0, 0, 0)$, then we can put coordinates on the projective plane as follows. Every projective point P is a line through O and some other point (p, q, r). Then every point on the line defining P is of the form $(\lambda p, \lambda q, \lambda r)$ for some λ. We write $P = [p : q : r]$, where the colons indicate that we only care about the ratios of the coordinates. If $\mathbb{A}^2$ is the plane $z = 1$, then the ordinary point on $\mathbb{A}^2$ corresponding to P is $(\frac{p}{r}, \frac{q}{r}, 1)$, or if we ignore the z-coordinate it is just $(\frac{p}{r}, \frac{q}{r})$. If $r = 0$, then P is an infinite point with *slope* $\frac{q}{p}$.

We can define projective coordinates for projective lines as well. A projective line l is defined by a single linear equation

$$dx + ey + fz = 0,$$

with not all of d, e, f equal to 0. Furthermore, this equation defines the same line if all of d, e, f are rescaled by the same nonzero λ. Thus we say that $l = (d : e : f)$. If $P = [p : q : r]$, then we have $P \in l$ if and only if

$$dp + eq + fr = 0.$$

The intersection of l with the ordinary plane $\mathbb{A}^2$ defined by $z = 1$ is just the line $dx + ey + f = 0$. The line at infinity has coordinates $(0 : 0 : 1)$.

The coordinate system described above can be called *cartesian projective coordinates*. There are other projective coordinate systems, one of the most useful of which is the *barycentric coordinate* system. In the barycentric coordinate system, a triangle ABC in $\mathbb{A}^2$ is fixed and the coordinates of three dimensional space are chosen such that $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$ - so the plane $\mathbb{A}^2$ is now defined by the equation $x+y+z = 1$. If P is an ordinary point in $\mathbb{A}^2$, then the projective coordinates $[p : q : r]$ of P are defined to be any three numbers p, q, r, not all zero, proportional to the three directed areas $[PBC], [APC], [ABP]$. In the barycentric coordinate system, a line $l = (d : e : f)$ is the set of points P such that

$$d[PBC] + e[APC] + f[ABP] = 0.$$

The line at infinity has barycentric coordinates $(1 : 1 : 1)$.

1.1 Cross Ratios

First we recall the definition of the ratio.

Definition 2. If A, B, C are three points on a line, not all equal, then we define their *ratio* to be

$$(A, B; C) = \frac{AC}{BC},$$

where the ratio is taken to be positive if the rays AC and BC point in the same direction, and negative otherwise. If l_1, l_2, l_3 are three directed lines passing through a point, not all equal, then their ratio is defined by

$$(l_1, l_2; l_3) = \frac{\sin \angle l_1 l_3}{\sin \angle l_2 l_3},$$

where the angles are oriented in the counterclockwise sense.

Exercise 1. (a) Show that if $A \neq B$ then there is a bijection between points C on the line AB and ratios $(A, B; C)$. Thus we can use the ratio as a coordinate on the line AB.

(b) Show that the ratio $(l_1, l_2; l_3)$ does not depend on the orientation of line l_3. Show that if $l_1 \neq l_2$ we can use the ratio $(l_1, l_2; l_3)$ as a coordinate on the set of lines through the point $l_1 \cap l_2$.

Exercise 2. Suppose that points A, B, C, not all equal, are on a line, and that point P is not on that line. Show that

$$\frac{(A, B; C)}{(PA, PB; PC)} = \frac{|PA|}{|PB|}.$$

Definition 3. If A, B, C, D are four points on a line, no three of them equal, then we define their *cross ratio* to be

$$(A, B; C, D) = \frac{(A, B; C)}{(A, B; D)} = \frac{AC}{CB}\Big/\frac{AD}{DB}.$$

If l_1, l_2, l_3, l_4 are four lines passing through a point, no three of them equal, then their cross ratio is defined by picking an orientation for each line, and then setting

$$(l_1, l_2; l_3, l_4) = \frac{(l_1, l_2; l_3)}{(l_1, l_2; l_4)} = \frac{\sin \angle l_1 l_3}{\sin \angle l_3 l_2}\Big/\frac{\sin \angle l_1 l_4}{\sin \angle l_4 l_2}.$$

Theorem 1 (The fundamental theorem of cross ratios)**.** *If A, B, C, D are on a line, no three of them equal, and if E is a point not on that line, then*

$$(EA, EB; EC, ED) = (A, B; C, D).$$

We would like to extend the above definitions to any four points or lines in the projective plane. One way to do this is to make special definitions if one of A, B, C, D is an infinite point: for instance, if ∞ is the infinite point on line AB, then we have

$$(A, B; C, \infty) = (A, B; C) = -\frac{AC}{CB}.$$

Similarly, if all of A, B, C, D are infinite points with slopes a, b, c, d, then their cross ratio is

$$(a, b; c, d) = \frac{c-a}{b-c}\Big/\frac{d-a}{b-d}.$$

However, the best way to do this is to simply change perspectives to get a coordinate system where none of A, B, C, D is an infinite point. In other words, we find a new plane $\mathbb{A}'^2$ not passing through the observation point O, which intersects the four lines corresponding to the projective points OA, OB, OC, OD at some new points A', B', C', D'. Then for finite points A, B, C, D we have

$$(A, B; C, D) = (OA, OB; OC, OD) = (A', B'; C', D'),$$

so the cross ratio in the new coordinate system will be the same as the original cross ratio. If one of A, B, C, D is an infinite point we use this formula as the *definition* of the cross ratio.

To check your understanding, calculate the cross ratio of four parallel lines in terms of the distances between them (parallel lines intersect at the infinite point corresponding to their common slope).

Exercise 3. Let ABC be a triangle, let M be the midpoint of AC, and let N be a point on line BM such that AN is parallel to BC. Let P be any point on line AC, and let Q be the intersection of line BP with line AN. Use cross ratios to prove that

$$\frac{AQ}{QN} = \frac{1}{2}\frac{AP}{PM}.$$

Exercise 4. (a) Check that for any number λ we have $(\lambda, 1; 0, \infty) = \lambda$.

(b) Show that $(A, B; D, C) = \frac{1}{(A,B;C,D)}$.

(c) Show that $(A, C; D, B) = \frac{1}{1-(A,B;C,D)}$.

Exercise 5. (a) Show that if $A \neq B$ and $(A, B; C, X) = (A, B; C, Y)$ then $X = Y$.

(b) Show that if $(A, B; C, D) = 1$ then either $A = B$ or $C = D$.

(c) Show that if $A \neq B$, $C \neq D$, and $(A, B; C, D) = (A, B; D, C)$ then $(A, B; C, D) = -1$.

Definition 4. If $(A, B; C, D) = -1$, then the four points A, B, C, D are called *harmonic*. We also say that D is the *harmonic conjugate* of C with respect to A, B. Sometimes we say that A, B, C, D are harmonic when three of them are equal.

Example 1. (i) If M is the midpoint of AB and if ∞ is the infinite point along line AB, then $(A, B; M, \infty) = -1$.

(ii) If ABC is a triangle, and if X, Y are the feet of the internal and external angle bisectors through C, then $(A, B; X, Y) = -1$ by the angle bisector theorem.

(iii) We have $(1, -1; x, \frac{1}{x}) = -1$ and $(0, \infty; x, -x) = -1$ for any x.

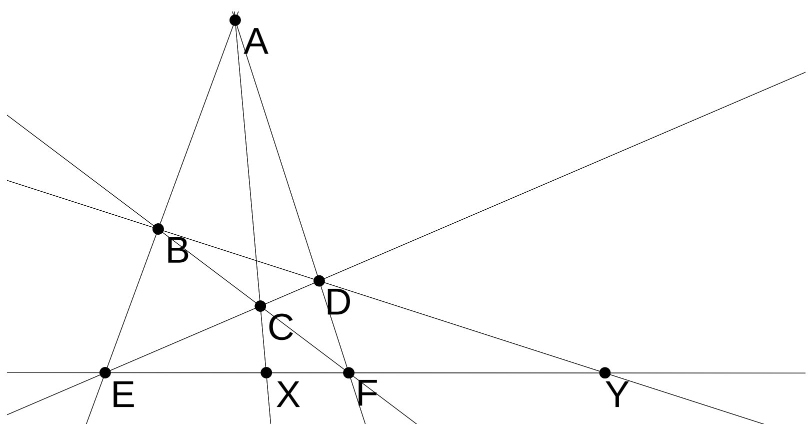

Figure 1: Quadrilateral Theorem

Theorem 2 (Quadrilateral Theorem)**.** *Let $ABCD$ be any quadrilateral. Let E be the intersection of sides AB and CD, and let F be the intersection of sides BC and DA. Let X be the intersection of diagonal AC with the line EF, and let Y be the intersection of diagonal BD with line EF. Then*

$$(E, F; X, Y) = -1.$$

Proof 1, using Ceva and Menelaus. By Ceva applied to triangle AEF and point C, we have

$$\frac{AB}{BE}\frac{EX}{XF}\frac{FD}{DA} = 1.$$

By Menelaus applied to triangle AEF and line BD, we have

$$\frac{AB}{BE}\frac{EY}{YF}\frac{FD}{DA} = -1.$$

Dividing these two equations, we get $(E, F; X, Y) = -1$. □

Proof 2, using cross ratios. Let P be the intersection of the diagonals AC and BD. We have

$$(E, F; X, Y) = (AE, AF; AX, AY) = (B, D; P, Y) = (CB, CD; CP, CY) = (F, E; X, Y).$$

Since $E \neq F$ and $X \neq Y$, we conclude that $(E, F; X, Y) = -1$. □

If EA, EB, EC, ED intersect a line l at points A', B', C', D', it often saves space to abbreviate the inference

$$(A, B; C, D) = (EA, EB; EC, ED) = (A', B'; C', D')$$

by just writing

$$(A, B; C, D) \overset{E}{=} (A', B'; C', D').$$

Now let's use this notation to give a compact proof of Desargues' Theorem:

Theorem 3 (Desargues' Theorem)**.** *Suppose that triangles ABC and XYZ are* perspective from a point*, that is, suppose that the lines AX, BY, CZ all meet at a point P. Then the triangles ABC and XYZ are* perspective from a line*, that is, the intersections $AB \cap XY$, $BC \cap YZ$, $CA \cap ZX$ all lie on a line.*

Proof. Let $U = BC \cap YZ$, $V = CA \cap ZX$, $W = AB \cap XY$. We want to show that U, V, W lie on a line, so we may as well suppose that $V \neq W$. Let Q, M, N be the intersections of line BY with the lines WV, AC, XZ, respectively. Then we have

$$(W, V; Q, BC \cap VW) \overset{B}{=} (A, V; M, C) \overset{P}{=} (X, V; N, Z) \overset{Y}{=} (W, V; Q, YZ \cap VW).$$

Thus $BC \cap VW = YZ \cap VW$, so the three lines BC, YZ, VW meet at the point U. □

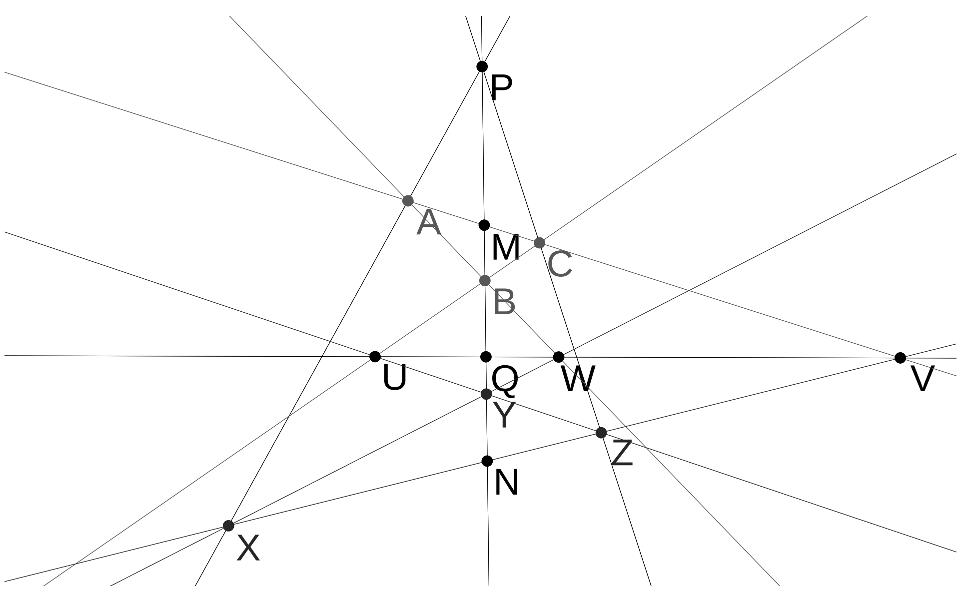

Figure 2: Desargues' Theorem

Exercise 6 (Pappus's Hexagon Theorem). Let A, B, C be on a line, and let D, E, F be on another line. Let $X = AE \cap BD, Y = BF \cap CE, Z = CD \cap AF$. Use cross ratios to show that X, Y, Z are on a line. (Hint: let $P = CD \cap BF$, and show that $(C, D; P, Z) = (C, D; P, CD \cap XY)$.)

Theorem 4 (Cross Ratio Equality)**.** *Let A, B, C, D be on a line, and let E, F, G, H be on another line. Let $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$. Then X, Y, Z are on a line if and only if $(A, B; C, D) = (E, F; G, H)$.*

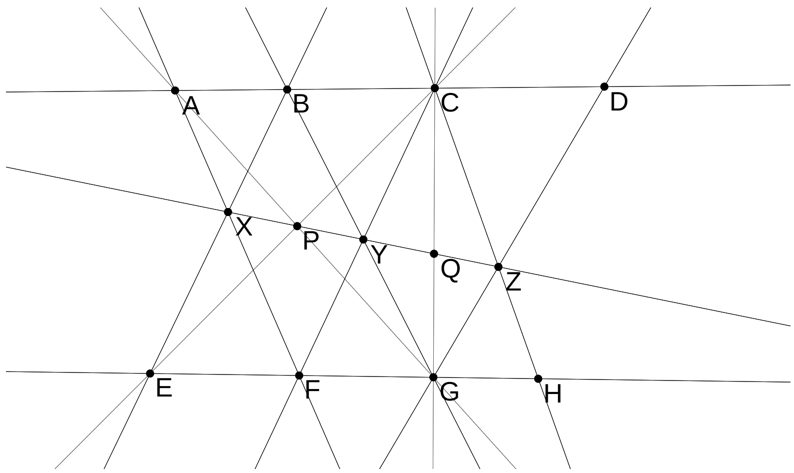

Figure 3: Equal cross ratios

Proof. Let $P = AG \cap CE, Q = CG \cap XY$. By Pappus's Theorem, P is on line XY. Projecting through G, we have $(A, B; C, D) \stackrel{G}{=} (P, Y; Q, DG \cap XY)$, and projecting through C, we have $(E, F; G, H) \stackrel{C}{=} (P, Y; Q, CH \cap XY)$. Thus $(A, B; C, D) = (E, F; G, H)$ if and only if CH, DG, and XY meet at a point. □

1.2 Cross Ratios on a Conic Section

Proposition 1. *Suppose that A, C, B, D are on circle ω, and that the (directed) arcs AC, CB, BD, DA of ω have central angles $2\alpha, 2\beta, 2\gamma, 2\delta$. Let E be any other point on ω. Then*

$$(EA, EB; EC, ED) = -\frac{\sin\alpha}{\sin\beta} \Big/ \frac{\sin\delta}{\sin\gamma}.$$

In particular, we have

$$|(EA, EB; EC, ED)| = \frac{|AC||BD|}{|AD||BC|}.$$

Corollary 1. *Let ω be any* conic section*, that is, any intersection of a cone $\mathcal{C}$ through the observation point O with the plane $\mathbb{A}^2$. If A, B, C, D, E, F are any six points on ω, then we have*

$$(EA, EB; EC, ED) = (FA, FB; FC, FD).$$

Proof. First we prove it when ω is a circle. By Proposition 2, we have

$$(EA, EB; EC, ED) = -\frac{\sin\alpha}{\sin\beta} \Big/ \frac{\sin\delta}{\sin\gamma} = (FA, FB; FC, FD).$$

For the general case, we choose another plane $\mathbb{A}'^2$ such that $\mathcal{C} \cap \mathbb{A}'^2$ is a circle. Let $A', B', ...$ be the intersections of lines $OA, OB, ...$ with the plane $\mathbb{A}'^2$. Then we have

$$(EA, EB; EC, ED) \stackrel{O}{=} (E'A', E'B'; E'C', E'D') = (F'A', F'B'; F'C', F'D') \stackrel{O}{=} (FA, FB; FC, FD).$$

□

Definition 5. If A, B, C, D are four points on a conic section ω, then we define the cross ratio of A, B, C, D with respect to ω by choosing any fifth point E on ω and setting

$$(A, B; C, D)_\omega = (EA, EB; EC, ED).$$

By Corollary 2, this doesn't depend on the choice of E.

Our first application of the cross ratio on a conic is to give a short proof of Pascal's theorem.

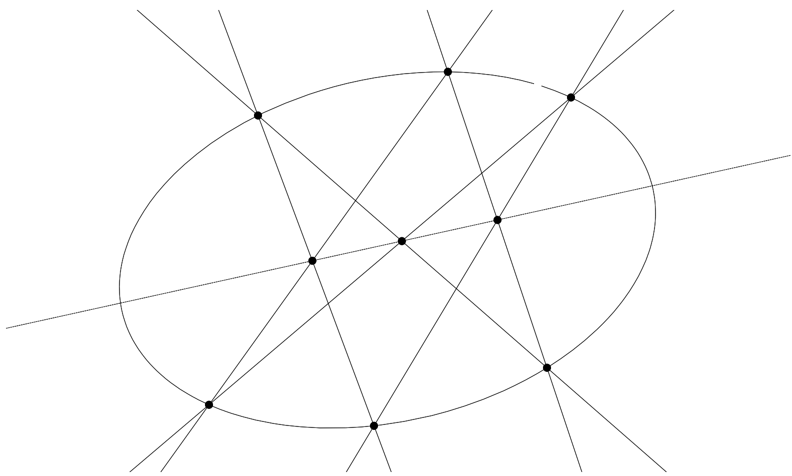

Figure 4: Pascal's Theorem

Theorem 5 (Pascal's Theorem)**.** *If $ABCDEF$ is any hexagon with vertices lying on a conic ω, then the three intersections of opposite sides $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ lie on a line.*

Proof. Let $L = BC \cap EF$, $M = CD \cap FA$, $N = AB \cap DE$ be the intersections of opposite sides of the hexagon. Let $P = AF \cap BC$ and $Q = AB \cap CD$. Then

$$(C, L; P, B) \stackrel{F}{=} (C, E; A, B)_\omega \stackrel{D}{=} (Q, N; A, B) \stackrel{M}{=} (C, MN \cap BC; P, B).$$

Thus $L = MN \cap BC$, so L is on the line MN. □

Exercise 7. (a) Given points A, B, C, D, E and a line l through A construct, using only a straightedge, the second point of intersection F between the line l and the conic through the points A, B, C, D, E.

(b) Given points A, B, C, D, E construct, using only a straightedge, the line l which is tangent to the conic through the points A, B, C, D, E at A.

Exercise 8. Suppose points A, B, C, D, E, F, G, H lie on a conic ω. Let $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$. Show that $(A, B; C, D)_\omega = (E, F; G, H)_\omega$ if and only if X, Y, Z are on a line.

Another easy application is a short proof of the butterfly theorem.

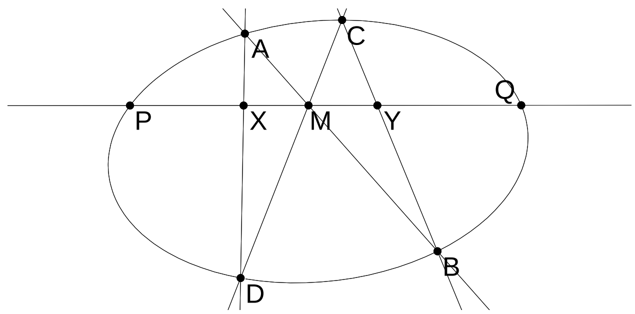

Figure 5: The Projective Butterfly Theorem

Theorem 6 (Projective Butterfly Theorem)**.** *Let ω be a conic, and let PQ be a chord on ω through the point M. Let AB and CD be two more chords of ω passing through M, and set $X = AD \cap PQ, Y = BC \cap PQ$. Then $(P, Q; M, X) = (Q, P; M, Y)$. In particular, if M is the midpoint of PQ then $|MX| = |MY|$.*

Proof.

$$(P, Q; M, X) \stackrel{A}{=} (P, Q; B, D)_\omega \stackrel{C}{=} (P, Q; Y, M) = (Q, P; M, Y).$$

We leave the proof of the last claim as an easy exercise to the reader. □

Definition 6. A cyclic quadrilateral $ACBD$ is called *harmonic* if $A \neq B, C \neq D$, and $|AC||BD| = |AD||BC|$.

Exercise 9. (a) Suppose P is a point outside circle ω. Let the two tangents from P to ω meet ω at A and B. Let l be a line through P meeting ω at two points C and D. Show that $ACBD$ is a harmonic quadrilateral.

(b) Let P, ω, A, B, C, D be as in (a), and let Q be the intersection of AB and CD. Show that $(C, D; P, Q) = -1$.

(c) Let P, ω, A, B be as in (a). Show that P', the inverse P with respect to ω, is on the line AB.

Exercise 10. Let ω be the unit circle, given in affine coordinates by the equation $x^2 + y^2 = 1$. Let $A = (1, 0), B = (0, 1), C = (-1, 0)$ in affine coordinates. Find the affine coordinates of the point D on ω such that $ACBD$ is a harmonic quadrilateral.

Exercise 11. Let $ABCDE$ be a regular pentagon inscribed in a circle ω. Compute $(A, B; C, D)_\omega$.

Exercise 12. Let A, B, C, D, E, F be six distinct points in the plane. Let $U = BC \cap DE, V = CA \cap EF, W = AB \cap FD, X = AB \cap EF, Y = BC \cap FD, Z = CA \cap DE$, so that hexagon $UZVXWY$ is the intersection of triangles ABC and DEF if it is convex. Show that the lines UX, VY, WZ meet in a point if and only if the points A, B, C, D, E, F lie on a conic.

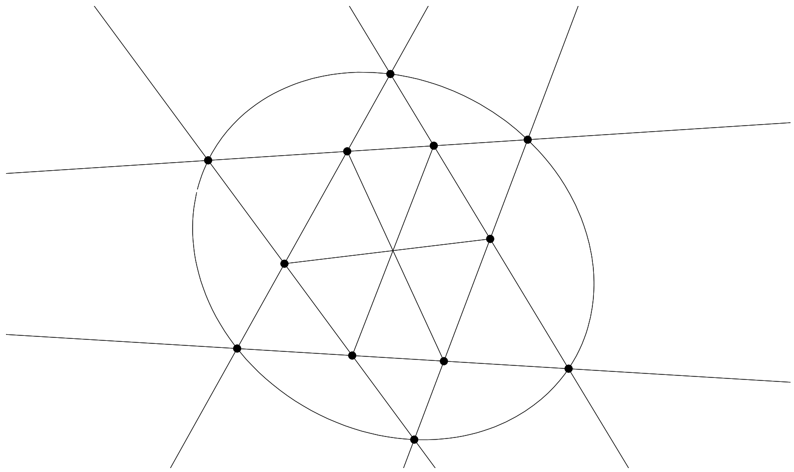

Figure 6: Exercise 12

1.3 Cross Ratios on the Inversive Plane

Just as we used three projective coordinates for the projective plane, we use two projective coordinates to describe a projective line. Specifically, the projective point $[s : t]$ will correspond to the ordinary point with coordinate $z = \frac{s}{t}$ if $t \neq 0$, and to the point ∞ if $t = 0$. When we allow s, t to be *complex* numbers, we get what is sometimes called the *complex projective line* $\mathbb{CP}^1$, the *inversive plane*, or the *Riemann sphere*.

We define cross ratios on the inversive plane the same way we define cross ratios on a line:

$$(a, b; c, d) = \frac{c-a}{b-c} \Big/ \frac{d-a}{b-d},$$

where now a, b, c, d are complex numbers corresponding to ordinary points A, B, C, D in the inversive plane.

Proposition 2. *The points A, B, C, D corresponding to the complex numbers a, b, c, d are on a circle or a line if and only if $(a, b; c, d)$ is a real number. If they are on a line, we have $(a, b; c, d) = (A, B; C, D)$, and if they are on a circle ω, we have $(a, b; c, d) = (A, B; C, D)_\omega$.*

Proof. Left as an exercise. □

Inversion around the unit circle is given by the simple formula $z \mapsto \frac{1}{\bar{z}}$ in the inversive plane. We have

$$\left(\frac{1}{\bar{a}}, \frac{1}{\bar{b}}; \frac{1}{\bar{c}}, \frac{1}{\bar{d}}\right) = \overline{\frac{\frac{1}{c} - \frac{1}{a}}{\frac{1}{b} - \frac{1}{c}} \Big/ \frac{\frac{1}{d} - \frac{1}{a}}{\frac{1}{b} - \frac{1}{d}}} = \overline{\frac{a-c}{c-b} \Big/ \frac{a-d}{d-b}} = \overline{(a, b; c, d)},$$

so inversion takes cross ratios to their complex conjugates. As a consequence, we see that inversion takes circles and lines to circles and lines, and furthermore it takes harmonic quadrilaterals to harmonic quadrilaterals.

Definition 7. To every two by two matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $ad - bc$ not equal to zero, we associate a transformation f_M of the inversive plane as follows. In projective coordinates

$[s:t]$, we write

$$f_M([s:t]) = [as+bt : cs+dt].$$

In ordinary coordinates $z = \frac{s}{t}$, we write

$$f_M(z) = \frac{az+b}{cz+d}.$$

The maps f_M are called *Möbius Transformations.*

Exercise 13. Show that for any two by two matrix M with nonzero determinant, and for any four points a, b, c, d on the inversive plane, we have

$$(f_M(a), f_M(b); f_M(c), f_M(d)) = (a, b; c, d).$$

Exercise 14. Check that composition of Möbius transformations corresponds to matrix multiplication, i.e. that for any two matrices M, N and any point $[s:t]$ we have

$$f_M(f_N([s:t])) = f_{MN}([s:t]).$$

Exercise 15. Let A, B, C, X, Y, Z be six points on the projective line, no two of A, B, C equal and no two of X, Y, Z equal. Prove that there is a Möbius transformation f such that $f(A) = X, f(B) = Y, f(C) = Z$.

1.4 Invertible functions on the line

Suppose a projective line $\mathbb{P}^1$ has coordinate z, and we have defined an invertible map $f : \mathbb{P}^1 \to \mathbb{P}^1$ via some geometric procedure that has no "configuration issues" (so for instance, taking the *leftmost* intersection of a circle with a line would not count). Since any geometrically defined map can be described algebraically by writing every point out in coordinates, our function f may be written as an algebraic function of z, and if there are no "configuration issues", then f must be a rational function, i.e. a ratio of two polynomials:

$$f(z) = \frac{p(z)}{q(z)}.$$

Since f is invertible, the equation $f(z) = w$ should have exactly one solution, so the polynomial

$$p(z) - wq(z)$$

should have degree 1 for every constant w. Thus p and q are both linear polynomials, and we can write

$$f(z) = \frac{az+b}{cz+d}.$$

Thus, f is in fact a Möbius transformation, and so f preserves the cross ratio. We record this as an informal theorem.

Theorem 7. *If f is an invertible function from a line to a line that is defined by a geometric procedure that has no "configuration issues", then f preserves the cross ratio. Furthermore, in this case f is a Möbius transformation.*

Exercise 16. Prove the converse: if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is any function that preserves the cross ratio, prove that f is a Möbius transformation, and find a geometric construction of the function f.

As an application, we consider the harmonic conjugation map. For any points A, B on $\mathbb{P}^1$, we define

$$h_{A,B}(C) = D \text{ if } (A, B; C, D) = -1.$$

We can construct D geometrically using the Quadrilateral Theorem, and $h_{A,B}$ is clearly invertible, so by the above discussion $h_{A,B}$ is a Möbius transformation. In coordinates, if A has coordinate a and B has coordinate b, we have

$$h_{a,b}(z) = \frac{(a+b)z - 2ab}{2z - a - b}.$$

Harmonic conjugation has the property that $h_{A,B}(h_{A,B}(C)) = C$ - in other words, harmonic conjugation is always an *involution*. In fact, this property characterizes harmonic conjugation.

Theorem 8. *If f is a Möbius transformation with the further property that f is an involution, i.e. $f(f(C)) = C$ for all points C, then f is either the identity map or there is a pair of (possibly imaginary) points A, B such that $f = h_{A,B}$.*

Proof. In coordinates, the equation $f(z) = z$ becomes a quadratic after clearing the denominator. If f is not the identity map, this quadratic will have two solutions, corresponding to two distinct points A, B. For any point C, write $D = f(C)$. Since f preserves the cross ratio, we have

$$(A, B; C, D) = (f(A), f(B); f(C), f(D)) = (A, B; D, C),$$

so the points A, B, C, D are harmonic. □

1.5 Angles and the circle points

Two special points in the projective plane allow us to talk about angles using cross ratios. These points are both infinite and imaginary, but we can treat them the same way we treat any other points in projective geometry. This allows us to solve many problems that are traditionally thought to be out of the scope of projective geometry.

Definition 8. The *circle points* are the points $\wp = [i : 1 : 0]$ and $\bar{\wp} = [1 : i : 0]$. These are the points at infinity of slope $-i$ and i.

Theorem 9 (Angle Theorem)**.** *If lines l, m intersect the line at infinity in points L, M, then*

$$(L, M; \wp, \bar{\wp}) = e^{2i\angle lm}.$$

In particular, lines l and m are orthogonal if and only if points $L, M, \wp, \bar{\wp}$ are harmonic.

Proof. Let s be the slope of line l and let t be the slope of line m. By the tangent subtraction formula, we have

$$\tan(\angle lm) = \frac{t - s}{1 + st}.$$

We have

$$\begin{aligned}
(L, M; \mathfrak{o}, \bar{\mathfrak{o}}) &= (s, t; -i, i) \\
&= \frac{s+i}{-i-t} \Big/ \frac{s-i}{i-t} \\
&= \frac{(s+i)^2(t-i)^2}{(s^2+1)(t^2+1)} \\
&= \frac{(st+1)^2 - (t-s)^2 + 2i(t-s)(st+1)}{(st+1)^2 + (t-s)^2} \\
&= \frac{1-\tan^2(\angle lm)}{1+\tan^2(\angle lm)} + i\frac{2\tan(\angle lm)}{1+\tan^2(\angle lm)} \\
&= \cos(2\angle lm) + i\sin(2\angle lm) \\
&= e^{2i\angle lm}.
\end{aligned}$$

□

Theorem 10. *A conic ω is a circle if and only if it passes through the two circle points.*

Proof. First, suppose ω is a circle with center (a, b) and radius r. In projective coordinates, ω is the set of points $[x : y : z]$ such that

$$(x - az)^2 + (y - bz)^2 = (rz)^2.$$

Plugging in, we can check that $[x : y : z] = [i : 1 : 0]$ and $[x : y : z] = [1 : i : 0]$ satisfy the equation defining ω.

Now suppose ω is any conic passing through $\mathfrak{o}, \bar{\mathfrak{o}}$. Let A, B, C, D be any four points on ω. Then we have

$$e^{2i\angle CAD} = (AC, AD; A\mathfrak{o}, A\bar{\mathfrak{o}}) = (C, D; \mathfrak{o}, \bar{\mathfrak{o}})_\omega = (BC, BD; B\mathfrak{o}, B\bar{\mathfrak{o}}) = e^{2i\angle CBD},$$

so the directed angles $\angle CAD$ and $\angle CBD$ are congruent modulo π. Thus A, B, C, D are concyclic.

□

Corollary 2. *Let A, B be two points on a circle ω with center O. Then*

$$(A, B; \mathfrak{o}, \bar{\mathfrak{o}})_\omega = e^{i\angle AOB}.$$

In particular, if A, B are diametrically opposite then $A, B, \mathfrak{o}, \bar{\mathfrak{o}}$ are harmonic.

Exercise 17. Say that four distinct points A, B, C, D on a line are *melodic* if we have

$$(A, B; C, D) = (A, D; B, C),$$

and make a similar definition for four points on a conic. Let $ABCDEF$ be a regular hexagon inscribed in a circle ω. Prove that the four points $A, B, \mathfrak{o}, \bar{\mathfrak{o}}$ are melodic with respect to ω.

1.6 Polar maps

Definition 9. We say that two points P, Q are *harmonic conjugates* with respect to a conic ω if P, Q, X, Y are harmonic, where X, Y are the (possibly imaginary) points of intersection of ω and PQ.

Theorem 11. *Let P be a point and ω a conic. Then the locus p of harmonic conjugates of P with respect to ω is a line.*

Proof 1, using tangents. Let U, V be the feet of the two tangents from P to ω. We will show that every point Q on the line UV is a harmonic conjugate of P with respect to ω. Let the line PQ meet ω at X, Y.

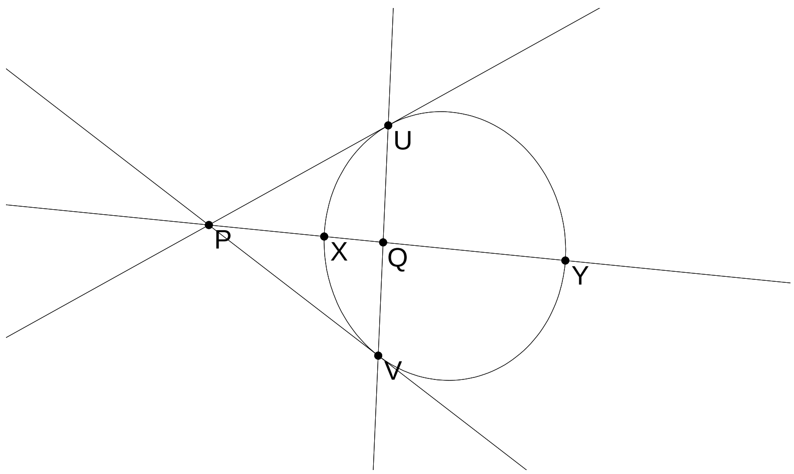

Figure 7: Proving P, Q are conjugate

Chasing cross ratios, we have

$$(P, Q; X, Y) \overset{U}{=} (U, V; X, Y)_\omega \overset{V}{=} (Q, P; X, Y),$$

so P, Q, X, Y are harmonic. $\square$

Proof 2, using chords. Let AC and BD be any two chords of ω passing through P. Let $E = AB \cap CD, F = AD \cap BC$. We will show that every point Q on the line EF is a harmonic conjugate of P with respect to ω. Let $AP \cap EF = R$, let ω meet PQ at X, Y, and let $U = AB \cap PQ, V = CD \cap PQ$.

By the quadrilateral theorem applied to the quadrilateral $BEDF$, the points A, C, P, R are harmonic. Projecting through E, we see that the four points U, V, P, Q are harmonic. Furthermore, by the projective butterfly theorem we have

$$(X, Y; P, U) = (Y, X; P, V).$$

Now suppose that Q' is the harmonic conjugate of P with respect to X, Y. Then if $h_{PQ'}$ denotes harmonic conjugation with respect to P, Q' we have

$$(X, Y; P, U) = (h_{PQ'}(X), h_{PQ'}(Y); h_{PQ'}(P), h_{PQ'}(U)) = (Y, X; P, h_{PQ'}(U)),$$

so $h_{PQ'}(U) = V$. Thus U, V, P, Q' are harmonic, so in fact we have $Q = Q'$. $\square$

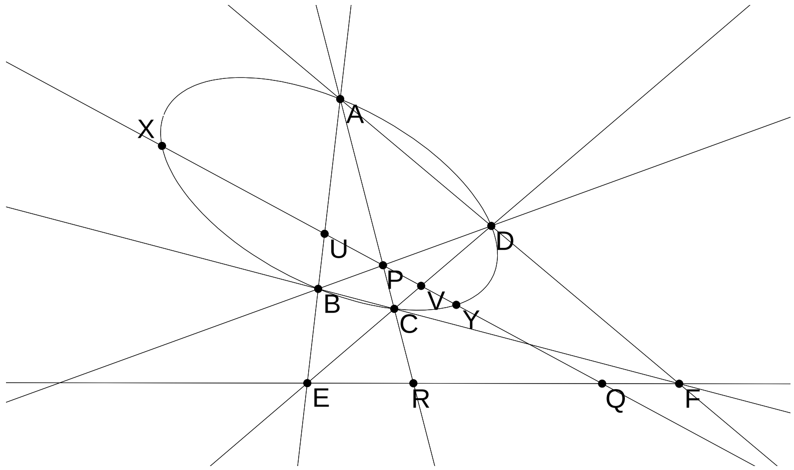

Figure 8: Proving P, Q are conjugate

Definition 10. If P is a point, ω a conic, and p is the locus of harmonic conjugates of P with respect to ω then we say that P is the *pole* of the line p, and p is the *polar* of the point P. When several conics are around, we will usually write ρ_ω for the *polar map* taking a point P to its polar p with respect to ω and taking a line p to its pole P with respect to ω.

Proposition 3. *Every line p has a unique pole P with respect to ω.*

Proof. Let Q, R be any two distinct points on p, and let their polars be q, r. Then q, r intersect in at least one point P. By definition, P is conjugate to Q and R with respect to ω, so the polar of P must be the line $QR = p$. Uniqueness is left as an exercise (consider the line joining two distinct poles of p). □

Proposition 4. *Let ω be a conic, let P, Q be points, and let p, q be their polars with respect to ω.*

(a) P is on q if and only if Q is on p.

(b) P is on p if and only if P is on ω, in which case p is tangent to ω.

(c) If X, Y are the feet of the tangent lines from P to ω, then $p = XY$.

Proof. The claims (a) and (b) are obvious from the definitions, while (c) follows easily from (a) and (b). □

Proposition 5. *Let ω be a conic. The either ω is a parabola or ω is centrally symmetric around a point O. If ω is a hyperbola, then O is the intersection of the asymptotes of ω.*

Proof. Let O be the pole of the line at infinity. If O is infinite, then ω must be tangent to the line at infinity at O, in which case ω is a parabola.

Now assume O is finite. Then for any chord X, Y through O, the points X, Y, O, and the point at infinity along XY are harmonic conjugates, so O is the midpoint of XY, i.e. ω is centrally symmetric around O. If ω is a hyperbola, then the asymptotes intersect at the pole of the line at infinity, which is O (this is still true if ω is an ellipse or a circle, but in that case the asymptotes have imaginary slopes). □

Theorem 12. *Let ω be a circle with center O. Let $P \neq O$ be a finite point, and let P' be its inverse with respect to the circle ω. Then the polar of P passes through P' and is perpendicular to the line OP.*

Proof. Let ω meet OP in the points X, Y. When we restrict inversion to the line OP, we see that it is a nontrivial involution fixing X and Y, so it must be harmonic conjugation with respect to X, Y. Thus X, Y, P, P' are harmonic conjugates (this can also be checked using coordinates, or alternatively by drawing tangents and using facts we have already proven about harmonic quadrilaterals).

Now let OP meet the line at infinity in the point L, and let M be the harmonic conjugate of L with respect to the circle points $\mathfrak{I}, \bar{\mathfrak{I}}$. Let l, m, o, p denote the polars of L, M, O, P, respectively. Since the circle points are the intersection of ω with the line at infinity, L and M are conjugate with respect to ω, so $M = o \cap l$ and thus $m = OL$. Since P is on m, M must be on p, so $p = MP'$. By the angle theorem, MP' is perpendicular to OP, so we are done. □

Theorem 13. *Let ω be a conic, and let points A, B, C, D on a line l have polars a, b, c, d. Then we have*

$$(A, B; C, D) = (a, b; c, d).$$

Proof. Note that all four lines a, b, c, d pass through L, the pole of l. First suppose that l is not tangent to ω. Let ω intersect l in points X, Y, and let a, b, c, d intersect l at the points A', B', C', D'. Then by the definition of the polar, the points A', B', C', D' are the harmonic conjugates of A, B, C, D with respect to X, Y. Thus if h_{XY} denotes harmonic conjugation with respect to X, Y, we have

$$(A, B; C, D) = (h_{XY}(A), h_{XY}(B); h_{XY}(C), h_{XY}(D)) = (A', B'; C', D') \stackrel{L}{=} (a, b; c, d).$$

Now suppose the line l is tangent to ω. Let M be any point not on l or ω, and let m be its polar with respect to ω. Then by the previous case applied to the line m,

$$(A, B; C, D) = (MA, MB; MC, MD) = (m \cap a, m \cap b; m \cap c; , m \cap d) = (a, b; c, d).$$ □

Exercise 18. Give a direct proof of Theorem 13 in the case that the line l is tangent to ω. (Hint: consider the map from l to ω taking a point P on l to the foot of the second tangent from P to ω. Prove that this map preserves the cross ratio.)

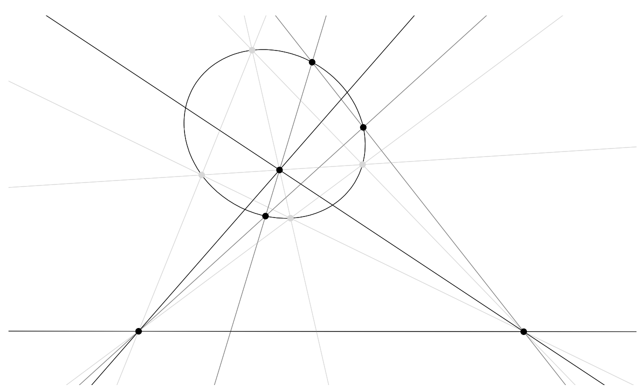

Figure 9: A self-polar triangle (Exercise 19)

Exercise 19. (a) Let $ABCD$ be a quadrilateral inscribed in a conic ω. Let $E = AB \cap CD, F = AD \cap BC$ be the intersections of the opposite sides, and let $G = AC \cap BD$ be the intersection of the diagonals. Prove that the triangle EFG is *self-polar* with respect to ω, that is, that the polars of E, F, G are FG, GE, EF, respectively.

(b) Let ABC be a self-polar triangle with respect to a conic ω, and let X, Y, Z be points on ω such that Z, A, Y are collinear and X, B, Z are collinear. Prove that Y, C, X are collinear.

Exercise 20. If a, b, c, d, e are lines tangent to a conic ω, define the cross ratio of a, b, c, d with respect to ω by

$$(a, b; c, d)_\omega = (a \cap e, b \cap e; c \cap e, d \cap e).$$

(a) Show that $(a, b; c, d)_\omega$ is independent of the choice of e.

(b) If a, b, c, d meet ω at A, B, C, D, show that

$$(a, b; c, d)_\omega = (A, B; C, D)_\omega.$$

Exercise 21 (Anders Kaseorg). Let ω, Ω be distinct circles, and let ρ_ω, ρ_Ω be the polar maps with respect to ω, Ω. Show that the composite map $\rho_\omega \circ \rho_\Omega \circ \rho_\omega \circ \rho_\Omega \circ \rho_\omega \circ \rho_\Omega$ is the identity if and only if the circles ω, Ω have equal radii and intersect in 60° arcs.

1.7 Coharmonic points

For any two pairs of distinct points $\{A, X\}$ and $\{B, Y\}$ on a line, we can find a Möbius transformation f satisfying $f(A) = X$, $f(X) = A$, $f(B) = Y$ (since Möbius transformations have three independent parameters). Since f preserves the cross ratio, for any other point C we must have

$$(X, A; f(C), C) = (A, X; C, f(C)) = (f(A), f(X); f(C), f(f(C))) = (X, A; f(C), f(f(C))),$$

so $C = f(f(C))$ and f is a harmonic conjugation in a pair of points $\{M, N\}$. Motivated by this fact, we make the following definition.

Definition 11. Three pairs of points $\{A, X\}, \{B, Y\}, \{C, Z\}$ on the same line are called *coharmonic* if there is another pair of (possibly imaginary) points $\{M, N\}$ such that

$$(M, N; A, X) = (M, N; B, Y) = (M, N; C, Z) = -1.$$

Theorem 14 (Main theorem of coharmonic points)**.** *Let A, B, C, X, Y, Z be on a line, no three the same, and suppose $A \neq X$. The following are equivalent:*

(a) *The three pairs of points $\{A, X\}, \{B, Y\}, \{C, Z\}$ are coharmonic.*

(b) *There is a Möbius transformation f satisfying $f(A) = X, f(B) = Y, f(C) = Z$ which is an involution.*

(c) $(A, X; B, C) = (X, A; Y, Z)$.

(d) $\frac{AY}{YC}\frac{CX}{XB}\frac{BZ}{ZA} = -1$.

Proof. By the above discussion, (a) and (b) are clearly equivalent. To see the equivalence of (b) and (c), let f be the Möbius function satisfying $f(A) = X, f(X) = A, f(B) = Y$. Then since f preserves the cross ratio, we have

$$(A, X; B, C) = (f(A), f(X); f(B), f(C)) = (X, A; Y, f(C)),$$

so $f(C) = Z$ if and only if $(A, X; B, C) = (X, A; Y, Z)$.

Now we show that (b) implies (d). We start by making the definition

$$(A, B, C; X, Y, Z) = \frac{AY}{YC}\frac{CX}{XB}\frac{BZ}{ZA}.$$

This can also be written as

$$(A, B, C; X, Y, Z) = -(A, C; Y, B)(B, A; Z, C)(C, B; X, A),$$

so it is preserved by any Möbius transformation. Thus

$$(A, B, C; X, Y, Z) = (f(A), f(B), f(C); f(X), f(Y), f(Z)) = (X, Y, Z; A, B, C) = 1/(A, B, C; X, Y, Z),$$

so $(A, B, C; X, Y, Z) = \pm 1$. To determine whether it is 1 or -1, we need to work with coordinates. Since $(A, B, C; X, Y, Z)$ is a projective invariant, we can choose coordinates so that the fixed points of f are 0 and ∞. Then $f(z) = -z$ for any z. Let the coordinates of A, B, C be a, b, c so the coordinates of X, Y, Z are $-a, -b, -c$. Then

$$(A, B, C; X, Y, Z) = \frac{a + b}{-b - c} \cdot \frac{c + a}{-a - b} \cdot \frac{b + c}{-c - a} = -1.$$

Finally, to see that (d) implies (b), note that for any A, B, C, X, Y there is a unique Z such that $(A, B, C; X, Y, Z) = -1$, and if f is a Möbius involution taking A to X and B to Y, then $(A, B, C; X, Y, f(C)) = -1$ by the above. □

Theorem 15 (Three Conic Law)**.** *Let A, B, C, D be any four points, no three on a line. Let l be a line passing through at most one of A, B, C, D. Let $\omega_1, \omega_2, \omega_3$ be three (possibly degenerate) conics passing through A, B, C, D. For each $i = 1, 2, 3$, let X_i, Y_i be the two points of intersection of conic ω_i with line l. Then the three pairs $\{X_1, Y_1\}, \{X_2, Y_2\}, \{X_3, Y_3\}$ are coharmonic.*

Proof. Consider the following map f from the line l to itself. For any point P on l, let ω_P be the conic passing through the points A, B, C, D, P, and define $f(P)$ to be the second point of intersection of ω_P with the line l. By Theorem 7, or more concretely by the solution to Exercise 7, f is a Möbius transformation. Since f is clearly also an involution satisfying $f(X_i) = Y_i$ for $i = 1, 2, 3$, the main theorem of coharmonic points shows that $\{X_1, Y_1\}, \{X_2, Y_2\}, \{X_3, Y_3\}$ are coharmonic. □

Exercise 22. (a) Let ω be a conic, and let P be a point not on ω, and let A, B, C be three points on ω. Let X, Y, Z be the second intersections of the lines PA, PB, PC with ω. Show that the three pairs $\{A, X\}, \{B, Y\}, \{C, Z\}$ are coharmonic with respect to the conic ω. (Hint: see Exercise 5.)

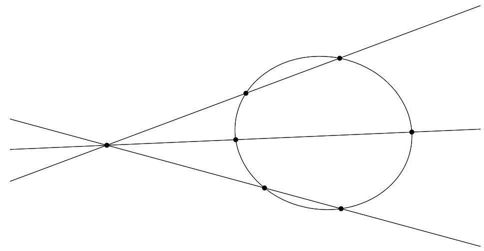

Figure 10: Coharmonic points on a conic

(b) Suppose that $ABCDEF$ is a convex hexagon inscribed in a circle ω. Show, using part (a), that the lines AD, BE, CF meet in a point if and only if

$$|AB||CD||EF| = |BC||DE||FA|.$$

(Hint: define $(A, E, C; D, B, F)_\omega$ for any conic ω, and calculate it in the special case that ω is a circle.) How is this related to the trigonometric form of Ceva's Theorem?

Exercise 23. Apply the Three Conic Law to give a second proof of the projective Butterfly Theorem: if ω is a conic, PQ is a chord on ω, M is a point on PQ, AB and CD are two more chords of ω passing through M, and $X = AD \cap PQ, Y = BC \cap PQ$, then $(P, Q; M, X) = (Q, P; M, Y)$. (Hint: show that $\{P, Q\}, \{M, M\}, \{X, Y\}$ are coharmonic.)

Exercise 24. Apply a degenerate case of the Three Conic Law to give a second proof of the Quadrilateral Theorem. (Hint: what does it mean for $\{X, Y\}, \{E, E\}, \{F, F\}$ to be coharmonic?)

Exercise 25. Apply the Three Conic Law to give a second proof of Desargues' Theorem. (Hint: In the notation of Theorem 3, show that $\{PA \cap VW, BC \cap VW\}, \{PB \cap VW, V\}, \{PC \cap VW, W\}$ are coharmonic, and compare the corresponding statement with A, B, C replaced by X, Y, Z.)

Exercise 26. Let ω, Ω be a pair of circles intersecting at points A, B, and let P be a point on the line AB. Let l be a line through P, let X, Y be the points of intersection between l and ω, and let U, V be the points of intersection between l and Ω. Show that

$$PX \cdot PY = PU \cdot PV.$$

Exercise 27. Let A, B, C, D, E lie on a conic ω, and let l be a line which is tangent to ω at E. Construct, using only a straightedge, the point $F \neq E$ on l such that the conic ω' passing through A, B, C, D, F is tangent to the line l at F.

Theorem 16 (Octagrammum Mysticum)**.** *Let A, B, C, D, E, F, G, H be eight points, no three on a line. Let $I = GH \cap BC, J = HA \cap CD, K = AB \cap DE$, etc., as in Figure 11. Then A, B, C, D, E, F, G, H lie on a conic if and only if I, J, K, L, M, N, O, P lie on a conic.*

Proof 1 (using coharmonicity). Suppose that I, J, K, L, M, N, O, P lie on a conic ω. It's enough to show that $(AF, AD; AH, AB) = (J, P; L, N)_\omega$, since then by symmetry we will have

$$(J, P; L, N)_\omega = (CF, CD; CH, CB) = (EF, ED; EH, EB) = (GF, GD; GH, GB),$$

from which we can conclude that C, E, G are on the conic through A, F, D, H, B. To this end, we project everything onto the line DF. Let $S = AB \cap DF, T = AH \cap DF, U = ML \cap DF, V =$

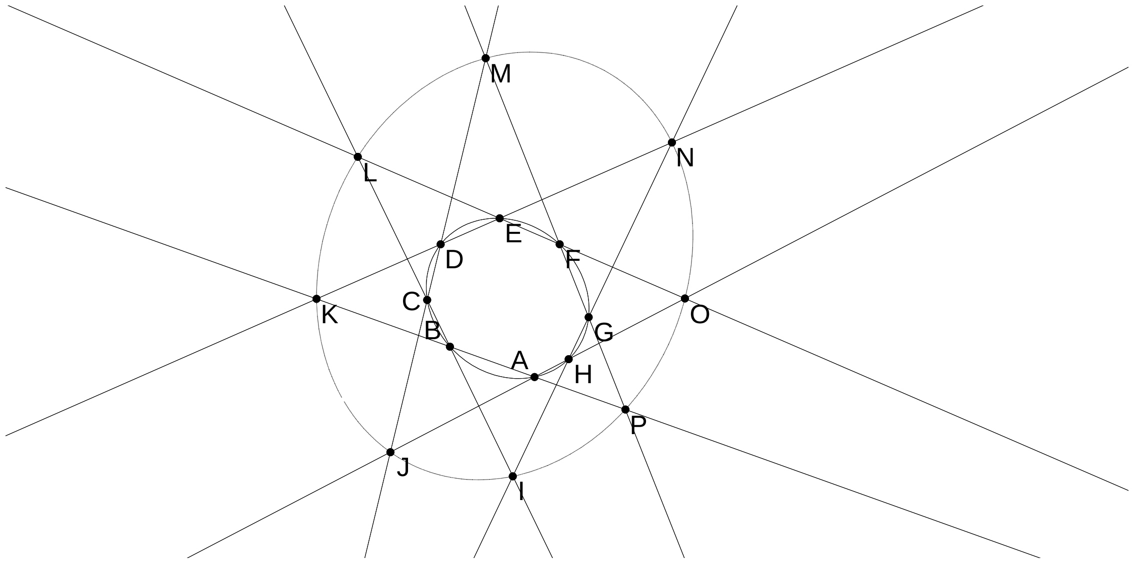

Figure 11: Octagrammum Mysticum

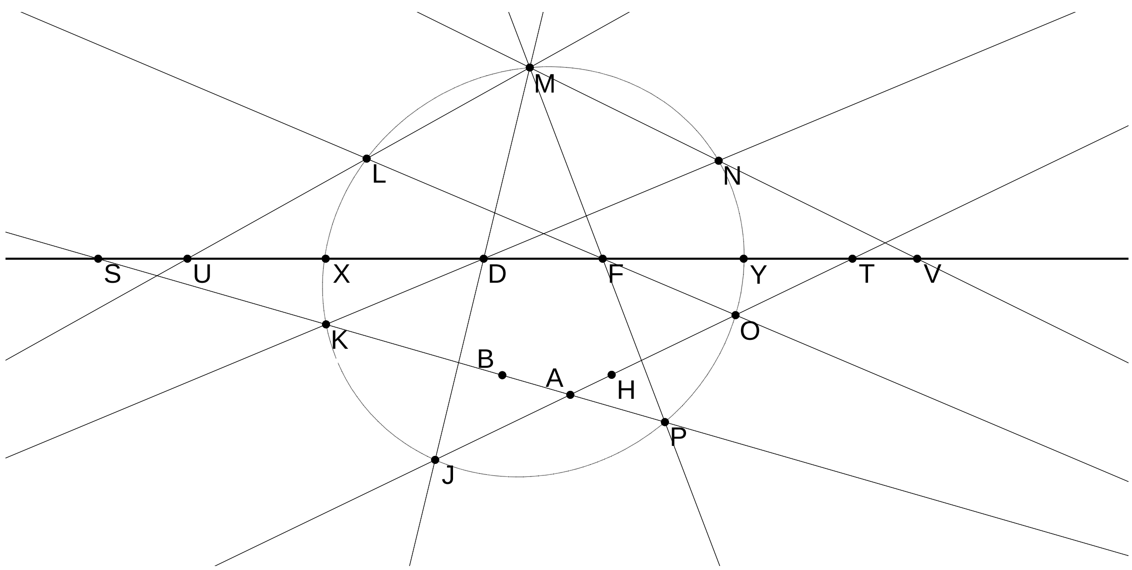

Figure 12: Proving $(AF, AD; AH, AB) = (J, P; L, N)_\omega$

$MN \cap DF$, and let X, Y be the (possibly imaginary) points of intersection between ω and DF. We have

$$(AF, AD; AH, AB) = (F, D; T, S)$$

and

$$(J, P; L, N)_\omega \overset{M}{=} (D, F; U, V),$$

so by Theorem 14 it's enough to show that $\{D, F\}, \{U, T\}, \{S, V\}$ are coharmonic.

Applying Three Conic Law to the points M, L, J, O, the line DF, the conic ω and the degenerate conics $ML \cup JO, MJ \cup LO$, we see that $\{D, F\}, \{X, Y\}, \{U, T\}$ are coharmonic. Similarly, applying the Three Conic Law to the points M, N, K, P, the line DF, the conic ω and the conics $MN \cup KP, MP \cup NK$, we see that $\{D, F\}, \{X, Y\}, \{S, V\}$ are coharmonic.

Thus the harmonic conjugation map that exchanges D with F and exchanges X with Y also exchanges U with T and S with V, so $\{D, F\}, \{U, T\}, \{S, V\}$ are coharmonic and we are done. □

Proof 2 (from [2], using Pascal's Theorem). Again, we assume that I, J, K, L, M, N, O, P lie on a conic ω. It's enough to show that G, H, A, B, C, D lie on a conic, since then by symmetry we have H, A, B, C, D, E on a conic, etc.

Let X be the intersection of lines KP and IJ. Applying Pascal's Theorem to the hexagon $MPKNIJ$ inscribed in the conic ω, we see that D, G, X lie on a line. From this we see that I, J, X are the intersections of the opposite sides of the hexagon $GHABCD$, so by the converse to Pascal's Theorem $GHABCD$ is also inscribed in a conic. □

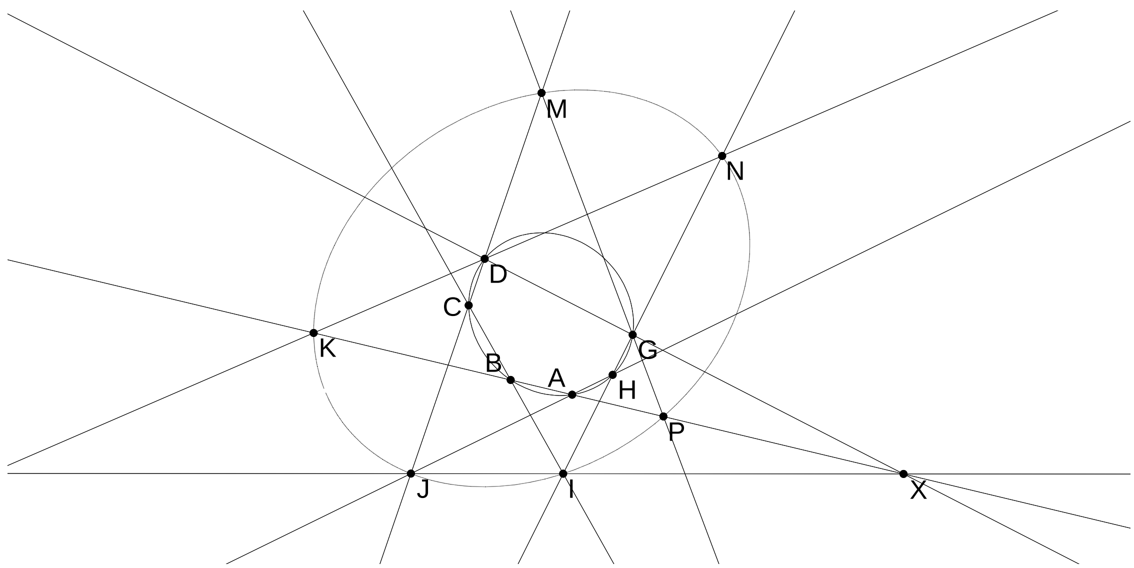

Figure 13: Applying Pascal

1.8 Symmetries of the plane

Definition 12. Let $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be a three by three matrix with nonzero determinant. The map $f_M : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $f([x : y : z]) = [ax + by + cz : dx + ey + fz : gx + hy + iz]$ is called a *projective transformation* of the plane.

Exercise 28. (a) Show that every projective transformation sends straight lines to straight lines, sends conics to conics, and preserves cross ratios.

(b) Show that if M, N are three by three matrices with nonzero determinants, then $f_M \circ f_N = f_{MN}$.

(c) Show that if A, B, C, D are any four points with no three on a line, and E, F, G, H are any four points with no three on a line, then there is a projective transformation f with $f(A) = E, f(B) = F, f(C) = G, f(D) = H$.

Definition 13. A bijection $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a *collineation* if it takes straight lines to straight lines.

Exercise 29. (a) Let A, B, C, D, E, F be six distinct points on a line. Show that $\{A, B\}, \{C, D\}, \{E, F\}$ are coharmonic if and only if

$$(A, B; C, D) = (A, B; C, E)(A, B; C, F).$$

(b) Given distinct points A, B, C, D, E on a line, construct points F and G on the same line such that

$$(A, B; C, F) = (A, B; C, D)(A, B; C, E)$$

and

$$(A, B; C, G) = (A, B; C, D) + (A, B; C, E)$$

using only a straightedge.

Exercise 30. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and such that for any $x, y \in \mathbb{R}$ we have $f(xy) = f(x)f(y)$ and $f(x + y) = f(x) + f(y)$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

Theorem 17 (Fundamental theorem of projective geometry)**.** *A bijection $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ is a collineation if and only if it is a projective transformation (here we write $\mathbb{RP}^2$ for the* real *points of the projective plane).*

Proof. We start by showing that if f is a collineation then it must preserve cross ratios. If A, B, C, D are distinct points on a line and E, F, G, H are distinct points on another line, then by Theorem 4 we can check whether $(A, B; C, D) = (E, F; G, H)$ by checking whether the points $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$ lie on a line. Since f is a collineation, we have $f(X) = f(AF) \cap f(BE) = f(A)f(F) \cap f(B)f(E)$ and so on, and $f(X), f(Y), f(Z)$ lie on a line if and only if X, Y, Z lie on a line, so

$$(A, B; C, D) = (E, F; G, H) \iff (f(A), f(B); f(C), f(D)) = (f(E), f(F); f(G), f(H)).$$

Thus we get a well-defined bijection $\tilde{f} : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ by taking

$$\tilde{f}((A, B; C, D)) = (f(A), f(B); f(C), f(D)).$$

This bijection automatically satisfies $\tilde{f}(0) = 0, \tilde{f}(1) = 1, \tilde{f}(\infty) = \infty$. By Exercise 29 we have $\tilde{f}(xy) = \tilde{f}(x)\tilde{f}(y)$ and $\tilde{f}(x + y) = \tilde{f}(x) + \tilde{f}(y)$ for any real x, y, and thus by Exercise 30 we must have $\tilde{f}(x) = x$ for all real x. Thus f preserves cross ratios.

To finish, note that by Exercise 28 we may assume without loss of generality that f fixes some collection of four points A, B, C, D such that no three are on a line. Letting $P = AB \cap CD$, we see that $f(P) = P$, and thus for any point X on AB we have

$$(A, B; P, X) = (f(A), f(B); f(P), f(X)) = (A, B; P, f(X)),$$

so $f(X) = X$. Thus if l is any line through C, and $X = l \cap AB$, then $f(l) = f(C)f(X) = CX = l$, so every line through C is sent to itself. Similarly, every line through A or B is sent to itself. Since any point E is determined by the three lines AE, BE, CE, every point E must be sent to itself, and we are done. $\square$

Remark 1. A collineation of $\mathbb{CP}^2$ might not preserve cross ratios: for instance, the map $[x : y : z] \mapsto [\bar{x} : \bar{y} : \bar{z}]$ taking every point to its complex conjugate sends every cross ratio to its complex conjugate. More generally, if $\tilde{f} : \mathbb{C} \to \mathbb{C}$ satisfies $\tilde{f}(1) = 1, \tilde{f}(xy) = \tilde{f}(x)\tilde{f}(y), \tilde{f}(x+y) = \tilde{f}(x)+\tilde{f}(y)$, then the map $[x : y : z] \mapsto [\tilde{f}(x) : \tilde{f}(y) : \tilde{f}(z)]$ is called an *automorphic collineation*, and sends a set of four points on a line with cross ratio c to a set of four points with cross ratio $\tilde{f}(c)$.

The same argument as above can be used to show that every collineation of $\mathbb{CP}^2$ can be written as the composition of an automorphic collineation and a projective transformation.

Definition 14. Let P be a point and l be a line not passing through P. Define the *projective reflection* $r_{P,l}$ by sending a point $Q \neq P$ to the harmonic conjugate of Q with respect to $P, PQ \cap l$ along the line PQ, and sending P to P.

Example 2. (a) Let l intersect the line at infinity at L. If P is on the line at infinity with $L, P, \wp, \bar{\wp}$ harmonic, then $r_{P,l}$ is (ordinary) reflection across the line l. As a consequence, (ordinary) reflections always interchange the two circle points.

(b) If l is the line at infinity, then $r_{P,l}$ is a $180°$ rotation around P (sometimes called a reflection through the point P).

Theorem 18. *For any point P and any line l not passing through P, the projective reflection $r_{P,l}$ is a projective transformation.*

Proof. We just need to show that $r_{P,l}$ sends lines to lines and preserves cross ratios. We leave this as an easy exercise to the reader. □

Definition 15. If A, B, C, D are four points with no three on a line and $\sigma : \{A, B, C, D\} \to \{A, B, C, D\}$ is a permutation, define r_σ to be the projective transformation taking A to $\sigma(A)$, B to $\sigma(B)$, etc. We will often write σ using its cycle decomposition, including the cycles of length 1, so that for instance $r_{(A)(B)(C\ D)}$ is the projective transformation taking A and B to themselves, and swapping C and D.

Exercise 31. Suppose A, B, C, D are four points with no three on a line.

(a) If $P = AB \cap CD$ and l is the line connecting $AC \cap BD$ to $AD \cap BC$, show that $r_{(A\ B)(C\ D)}$ is the projective reflection $r_{P,l}$.

(b) Show that if ω is a conic passing through A, B, C, D then $r_{(A\ B)(C\ D)}(\omega) = \omega$.

Exercise 32. (a) Show that for every permutation $\sigma : \{A, B, C, D\} \to \{A, B, C, D\}$ we can write r_σ as a composition of two projective reflections.

(b) Show that a projective transformation defined by a three by three matrix M can be written as a composition of two projective reflections if and only if the eigenvalues of M are in a geometric progression.

Exercise 33. Let $f(p, q, r), g(p, q, r), h(p, q, r)$ be homogeneous polynomials of the same degree having no common factor. The map $[p : q : r] \mapsto [f(p, q, r) : g(p, q, r) : h(p, q, r)]$ is called *biregular* if it is defined everywhere (i.e. f, g, h are never simultaneously 0 unless p, q, r are all 0) and is a bijection of the complex points of the projective plane. Prove that every biregular map is a projective transformation.

1.9 The Cross Cross Ratio

Since any four points (no three on a line) can be sent to any other four points (no three on a line) by a projective transformation, there are no interesting invariants of four general points in the plane. If we have five general points A, B, C, D, E, then we can form the cross ratio $(EA, EB; EC, ED)$. Going one step further, we have the following natural definition.

Definition 16. Let A, B, C, D, E, F be six points in the plane, such that either none of ACE, ADF, BCF, BDE are lines or none of ACF, ADE, BCE, BDF are lines. Define their *cross cross ratio* to be

$$(A, B; C, D; E, F) = \frac{(EA, EB; EC, ED)}{(FA, FB; FC, FD)}.$$

First we will prove that this definition is more symmetric than it seems.

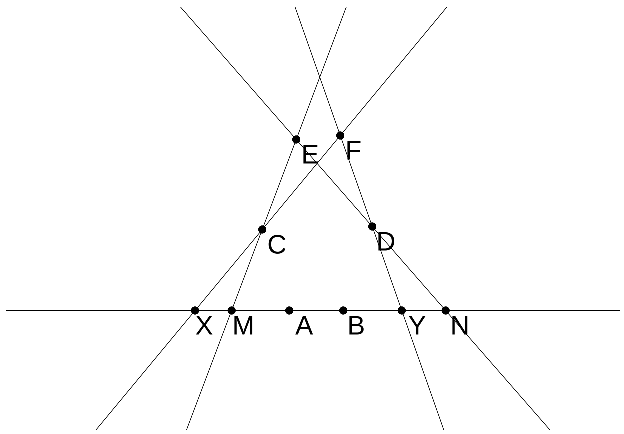

Figure 14: Symmetry of the cross cross ratio

Theorem 19. *Let A, B, C, D, E, F be as above. Then we have*

$$(A, B; C, D; E, F) = (A, B; E, F; C, D).$$

Proof. We start by projecting everything onto the line AB. Let $M = EC \cap AB, N = ED \cap AB, X = FC \cap AB, Y = FD \cap AB$. Then we have

$$\begin{aligned}
(A, B; C, D; E, F) &= \frac{(EA, EB; EC, ED)}{(FA, FB; FC, FD)} = \frac{(A, B; M, N)}{(A, B; X, Y)} \\
&= \frac{(A, B; M)}{(A, B; N)} \Big/ \frac{(A, B; X)}{(A, B; Y)} = \frac{(A, B; M)}{(A, B; X)} \Big/ \frac{(A, B; N)}{(A, B; Y)} \\
&= \frac{(A, B; M, X)}{(A, B; N, Y)} = \frac{(CA, CB; CE, CF)}{(DA, DB; DE, DF)} = (A, B; E, F; C, D).
\end{aligned}$$

□

Proposition 6. *Let two circles ω, ω' intersect at points A, B, and let C be a point on ω, D a point on ω'. Let θ be the (directed) angle of intersection between the circles ω, ω' at A. Then we have*

$$(A, B; \wp, \bar{\wp}; C, D) = e^{2i\theta}.$$

In particular, ω and ω' are orthogonal if and only if $(A, B; \wp, \bar{\wp}; C, D) = -1$.

Proof.

$$(A, B; \wp, \bar{\wp}; C, D) = \frac{(A, B; \wp, \bar{\wp})_\omega}{(A, B; \wp, \bar{\wp})_{\omega'}} = e^{2i(\angle ACB - \angle ADB)} = e^{2i\theta}.$$

□

Definition 17. If conics ω, ω' meet in points A, B, C, D, set

$$(A, B; C, D; \omega, \omega') = \frac{(A, B; C, D)_\omega}{(A, B; C, D)_{\omega'}}.$$

If $(A, B; C, D; \omega, \omega') = -1$, we say that the conics ω, ω' are *projectively orthogonal with respect to the partition* $\{A, B\}, \{C, D\}$ of their intersection points.

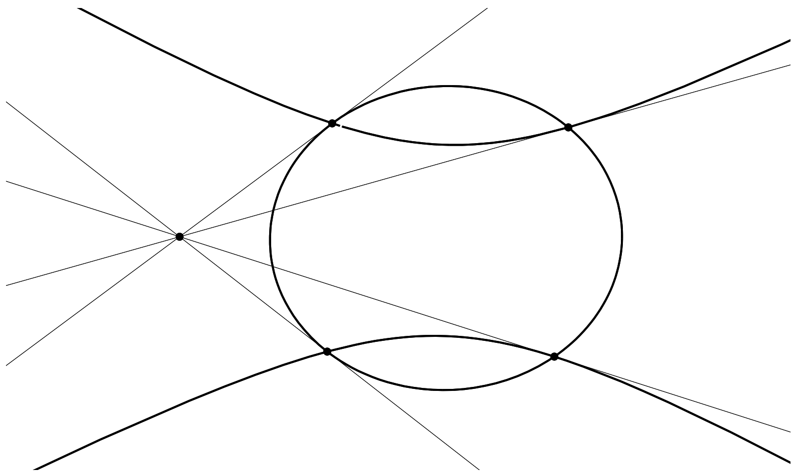

Figure 15: Projectively orthogonal conics

Theorem 20. *Two conics ω, Ω meeting in points A, B, C, D are projectively orthogonal with respect to the partition $\{A, B\}, \{C, D\}$ if and only if the two tangents to ω at A and B meet the two tangents to Ω at C and D.*

Proof. Let $E = AC \cap BD, F = AD \cap BC$. We will project everything onto the line EF: let $X = AB \cap EF$, let $Y = CD \cap EF$, let P be the intersection of the tangent to ω at A with EF, and let Q be the intersection of the tangent to Ω at C with EF.

Projecting through A or B, we have

$$(A, B; C, D)_\omega \stackrel{A}{=} (P, X; E, F) \stackrel{B}{=} (BP \cap \omega, A; D, C)_\omega,$$

so BP is also tangent to ω, and similarly we have

$$(A, B; C, D)_\Omega \stackrel{C}{=} (E, F; Q, Y) \stackrel{D}{=} (B, A; DQ \cap \Omega, C)_\Omega$$

and DQ is tangent to Ω. By the quadrilateral theorem, we have

$$(E, F; X, Y) = -1,$$

so

$$\frac{(A, B; C, D)_\omega}{(A, B; C, D)_\Omega} = \frac{(E, F; P, X)}{(E, F; Q, Y)} = \frac{(E, F; P, Q)}{(E, F; X, Y)} = -(E, F; P, Q).$$

Thus $(A, B; C, D; \omega, \Omega) = -1$ if and only if $P = Q$. □

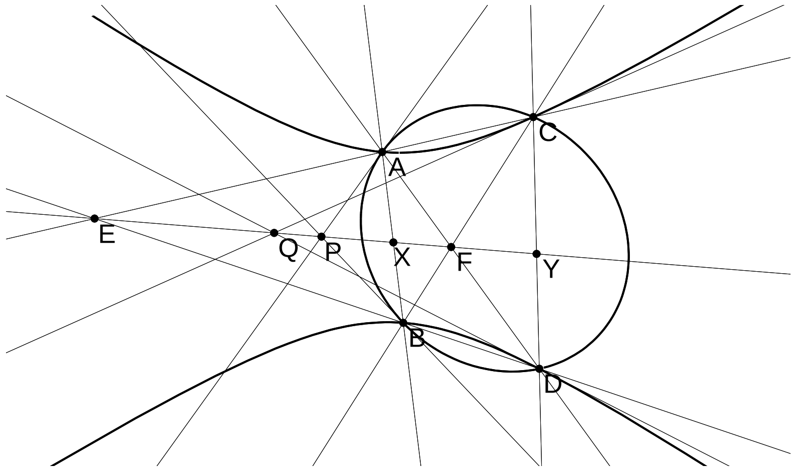

Figure 16: Checking orthogonality

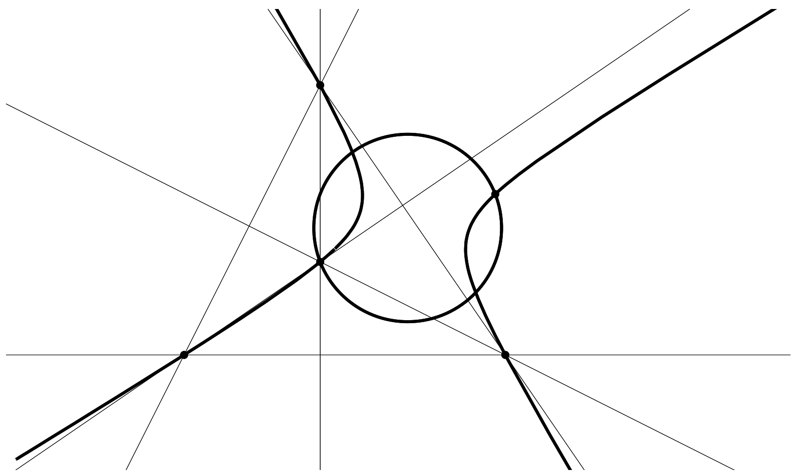

Figure 17: Exercise 34

Exercise 34. Let H be the orthocenter of triangle ABC, and let P be any point other than H. Let ω be the circle with diameter HP, and let Ω be the conic through A, B, C, H, P.

(a) Show that the asymptotes to Ω meet at a right angle.

(b) Show that if ω, Ω also meet at points X, Y, then ω is projectively orthogonal to Ω with respect to the partition $\{H, P\}, \{X, Y\}$.

Exercise 35. Suppose conics ω, Ω meet at A, B, C, D and are projectively orthogonal with respect to the partition $\{A, B\}, \{C, D\}$ of their intersection points. Let l be a line meeting ω at P, Q and meeting Ω at R, S.

(a) Show that $(P, Q; A, B)_\omega = -1$ if and only if $(R, S; C, D)_\Omega = -1$.

(b) Show that if $(P, Q; A, B)_\omega = -1$ then $(P, Q; R, S) = -1$.

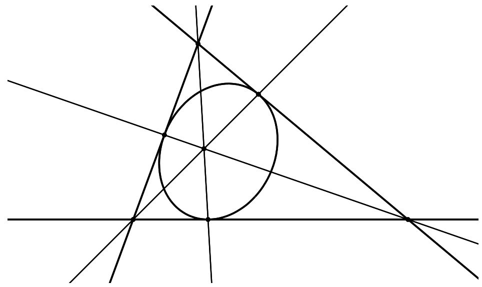

Figure 18: Exercise 36(a)

1.10 A few miscellaneous exercises

Exercise 36. (a) Let ABC be a triangle, let D be a point on BC, let E be a point on CA, and let F be a point of AB. Show that the lines AD, BE, CF meet in a point if and only if there is a conic ω which is tangent to BC at D, tangent to CA at E, and tangent to AB at F.

(b) Let ABC be a triangle and let P be a point not lying on any edge of ABC. Let U, X be points on BC with $X = r_{(A)(P)(B\,C)}(U)$, let V, Y be points on CA with $Y = r_{(B)(P)(A\,C)}(V)$, and let W, Z be points on AB with $Z = r_{(C)(P)(A\,B)}(W)$. Show that U, V, W, X, Y, Z lie on a conic.

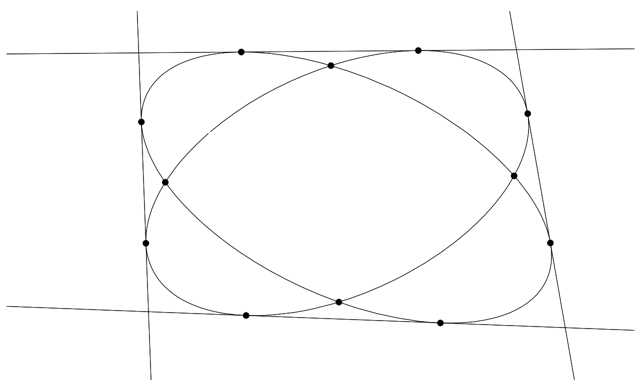

Figure 19: Exercise 37

Exercise 37. (a) Given points A, B, C, D and a line e, there are two conics ω, Ω passing through A, B, C, D and tangent to e. Construct the other three common tangent lines f, g, h to the conics ω, Ω using only the points A, B, C, D, the line e, and a straightedge.

(b) Show that if you order e, f, g, h correctly, you have

$$(A, B; C, D)_\omega = (e, f; g, h)_\Omega.$$

(c) Show that polar maps send projectively orthogonal pairs of conics to projective orthogonal pairs of conics.

Exercise 38. Suppose that A, B, C, D, E, F, G, H are eight distinct points in the plane such that the four lines AB, CD, EF, GH meet in a point, the four lines AC, BD, EG, FH meet in a point,

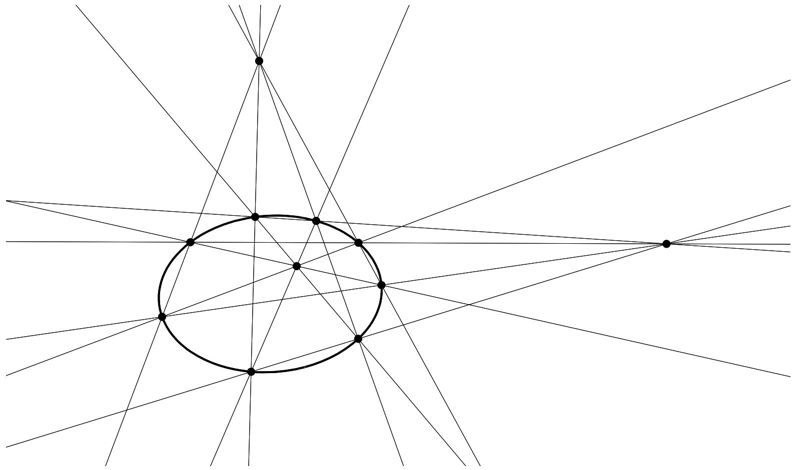

Figure 20: Exercise 38

and the four lines AD, BC, EH, FG meet in a point. Show that A, B, C, D, E, F, G, H all lie on a single conic.

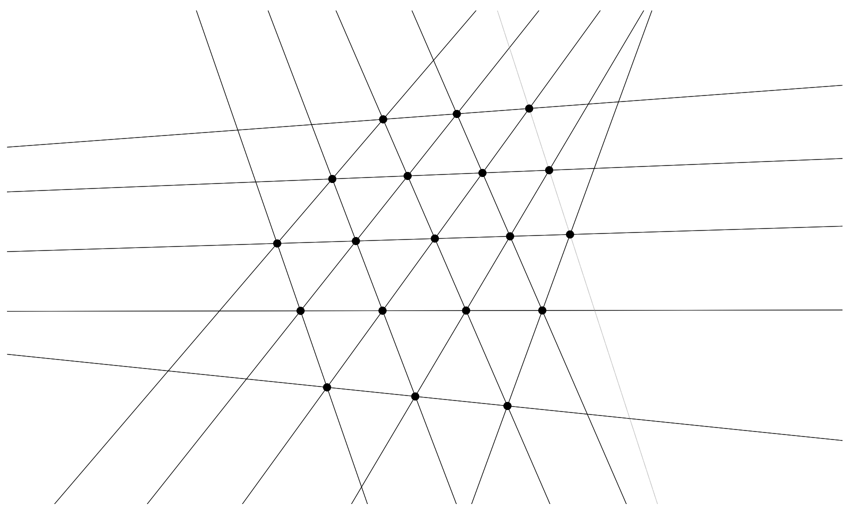

Figure 21: Triangular grid lemma

Exercise 39 (Triangular grid lemma). Let $a_1, a_2, a_3, a_4, b_1, b_2$ be six distinct lines. Let c_1 be the line through $a_3 \cap b_1$ and $a_2 \cap b_2$. Let c_2 be the line through $a_4 \cap b_1$ and $a_3 \cap b_2$. Let b_3 be the line through $a_1 \cap c_1$ and $a_2 \cap c_2$. Let c_3 be the line through $a_4 \cap b_2$ and $a_3 \cap b_3$. Let b_4 be the line through $a_1 \cap c_2$ and $a_2 \cap c_3$. Let c_4 be the line through $a_4 \cap b_3$ and $a_3 \cap b_4$. Let b_5 be the line through $a_1 \cap c_3$ and $a_2 \cap c_4$. Let c_5 be the line through $a_4 \cap b_4$ and $a_3 \cap b_5$. Show that the three points $b_1 \cap c_3, b_2 \cap c_4, b_3 \cap c_5$ are on a line. (Hint: use Theorem 4.)

2 Cross ratios in other geometries

2.1 Cremona involutions and blow ups

Let A, B, C, D be four points in the projective plane, no three on a line. Choose projective coordinates such that $A = [1:0:0], B = [0:1:0], C = [0:0:1], D = [1:1:1]$ (one way to do this is to start with barycentric coordinates on the triangle A, B, C, and then rescale the coordinates to make $D = [1:1:1]$). For future reference, let $E = [-1:1:1], F = [1:-1:1], G = [1:1:-1]$ in this coordinate system.

Exercise 40. Show that E, F, G satisfy $DE \cap FG = A, DF \cap EG = B, DG \cap EF = C$, and that they are uniquely determined by these conditions. Show that E is the harmonic conjugate of D with respect to $A, AD \cap BC$.

One of the simplest nonlinear functions we can write down is the Cremona involution: if p, q, r are all nonzero, it takes the point $P = [p:q:r]$ in the above coordinate system to the point $f_{ABCD}(P) = [\frac{1}{p} : \frac{1}{q} : \frac{1}{r}]$. We would like to extend this to an involution of the plane. Clearing denominators, we get $f_{ABCD}(P) = [qr : pr : pq]$, and this lets us define $f_{ABCD}(P)$ as long as no two of p, q, r are 0, i.e. as long as P is not equal to one of A, B, C. If P is on line BC, then $p = 0$, so $f_{ABCD}(P) = [qr : 0 : 0] = A$, and f_{ABCD} is not injective. We can fix these problems by "blowing up" the points A, B, C.

Definition 18. If A, B, C are three distinct points in the projective plane, we set

$$\mathrm{Bl}_{ABC}\mathbb{P}^2 = \{(P, l_A, l_B, l_C) \mid P \in \mathbb{P}^2, \{A, P\} \subset l_A, \{B, P\} \subset l_B, \{C, P\} \subset l_C\}.$$

If $(P, l_A, l_B, l_C) \in \mathrm{Bl}_{ABC}\mathbb{P}^2$ has $P \neq A, B, C$, then $l_A = AP, l_B = BP, l_C = CP$, and we write P as shorthand for (P, l_A, l_B, l_C). Let e_A be the set of points (P, l_A, l_B, l_C) in $\mathrm{Bl}_{ABC}\mathbb{P}^2$ with $P = A$, that is,

$$e_A = \{(A, l, AB, AC) \mid A \in l\},$$

and define e_B, e_C similarly. If $(A, l, AB, AC) \in e_A$, we write (A, l) as shorthand for it. If $P = (A, l)$, we write AP as shorthand for l. The three lines e_A, e_B, e_C are called the *exceptional lines* above A, B, C. We say that a curve ω passing through A intersects the exceptional line e_A in the point (A, l_A) if line l_A is tangent to ω at A.

In coordinates, we have

$$\mathrm{Bl}_{ABC}\mathbb{P}^2 = \{([p:q:r], (0:a:b), (c:0:d), (e:f:0)) \mid aq + br = cp + dr = ep + fq = 0\}.$$

Proposition 7. *The map $[p:q:r] \mapsto [\frac{1}{p} : \frac{1}{q} : \frac{1}{r}]$, defined for $p, q, r \neq 0$, extends to an involution $f_{ABCD} : \mathrm{Bl}_{ABC}\mathbb{P}^2 \to \mathrm{Bl}_{ABC}\mathbb{P}^2$. The extended involution f_{ABCD} takes e_A (resp. e_B, e_C) bijectively to BC (resp. AC, AB). The fixed points of f_{ABCD} are D, E, F, G, and we have $f_{ABCD} = f_{ABCE} = f_{ABCF} = f_{ABCG}$.*

Proof. In coordinates, if $P = ([p:q:r], (0:a:b), (c:0:d), (e:f:0))$ we set

$$f_{ABCD}(P) = \begin{cases} ([aq : ap : -bp], (0:b:a), (d:0:c), (f:e:0)) & \text{if } p \neq 0, \\ ([cq : cp : -dq], (0:b:a), (d:0:c), (f:e:0)) & \text{if } q \neq 0, \\ ([er : -fr : ep], (0:b:a), (d:0:c), (f:e:0)) & \text{if } r \neq 0. \end{cases}$$

Checking that this is well-defined, along with checking the other claims of the proposition, is left as an easy exercise to the reader. □

Remark 2. More generally, for any three homogeneous polynomials $f(p,q,r), g(p,q,r), h(p,q,r)$ of the same degree having no common factor we can define a map $[p:q:r] \to [f(p,q,r):g(p,q,r):h(p,q,r)]$, which is well-defined whenever f, g, h are not simultaneously zero. Such a map is called a *rational map*. It is called *birational* if it is usually one-to-one - in this case you can write down a rational function which inverts it whenever both are defined. Noether and Castelnuovo have proved that every birational map $\mathbb{P}^2 \to \mathbb{P}^2$ can be built out of projective transformations and Cremona involutions.

Proposition 8. *Let A, B, C, D, E, F, G be such that $A = DE \cap FG, B = DF \cap EG, C = DG \cap EF$, and suppose that $f_{ABCD}(P) = Q$. Then we have*

$$(AP, AQ; DE, FG) = (BP, BQ; DF, EG) = (CP, CQ; DG, EF) = -1.$$

In other words, AQ is the harmonic conjugate of AP with respect to AD, AF, and similarly for BQ, CQ.

Proof. By symmetry, it's enough to show that $(AP, AQ; DE, FG) = -1$. In the coordinate system described above, suppose that $AP = (0:a:b)$. We then have $DE = (0:1:-1), FG = (0:1:1), AQ = (0:b:a)$, so

$$(AP, AQ; DE, FG) = (a/b, b/a; -1, 1) = -1. \qquad \square$$

Example 3. Let G be the centroid of triangle ABC, and suppose $f_{ABCG}(P) = Q$. Let FED have parallel sides to ABC, such that A is the midpoint of DE, B is the midpoint of DF, and C is the midpoint of EF. Let $M = AG \cap BC, X = AP \cap BC, Y = AQ \cap BC, \infty = AD \cap BC$. Then

$$(X, Y; \infty, M) = (AP, AQ; DE, FG) = -1,$$

so X is the reflection of Y across M, the midpoint of BC. Similarly, $BP \cap AC$ is the reflection of $BQ \cap AC$ across the midpoint of AC, etc. The point Q is called the *isotomic conjugate* of P.

Corollary 3. *Let $f = f_{ABCD}$. For any four points $P, Q, R, S \in \mathrm{Bl}_{ABC}\mathbb{P}^2$ we have*

$$(AP, AQ; AR, AS) = (Af(P), Af(Q); Af(R), Af(S)).$$

Proof. Harmonic conjugation preserves the cross ratio. $\square$

Theorem 21. *Let l be a line which does not pass through any of A, B, C. Then $f_{ABCD}(l)$ is a* circumconic*, that is, a conic passing through all three of A, B, C. Conversely, if ω is a circumconic then $f_{ABCD}(\omega)$ is a line which does not pass through any of A, B, C.*

Proof. Write $f = f_{ABCD}$, and let P, Q, R be any three points on l. Let $S = l \cap BC$, so that $f(S) \in e_A$. By the Corollary, we have

$$(Bf(P), Bf(Q); Bf(R), BA) = (P, Q; R, S) = (Cf(P), Cf(Q); Cf(R), CA),$$

so $f(R)$ lies on the conic ω through $A, B, C, f(P), f(Q)$. The converse is left as an exercise. $\square$

Exercise 41. Let I be the incenter of triangle ABC. The map f_{ABCI} is called *isogonal conjugation.*

(a) Show that $f_{ABCI}(\eth) = \bar{\eth}$.

(b) Let Ω be the circumcircle of triangle ABC. Show that $f_{ABCI}(\Omega)$ is the line at infinity.

(c) Let m be the median through A. Show that $f_{ABCI}(m)$ passes through the pole of BC with respect to Ω. (Hint: show that the intersections of m, BC, AB, AC with the line at infinity are harmonic, then apply f_{ABCI}.)

(d) Let ω be the circumcircle of triangle BCI. Show that $f_{ABCI}(\omega) = \omega$.

Exercise 42. Write $f = f_{ABCD}$, let l be a line which doesn't pass through any of A, B, C, let $\omega = f(l)$, and let P, Q, R, S be any four points on l. Show that

$$(P, Q; R, S) = (f(P), f(Q); f(R), f(S))_\omega.$$

Theorem 22. *If ω is a conic which passes through B and C but not A, then $f_{ABCD}(\omega)$ is also a conic passing through B and C but not A. We have $f_{ABCD}(\omega) = \omega$ if and only if ω either passes through D and E or passes through F and G.*

Proof. Write $f = f_{ABCD}$, and let P, Q, R, S be any four points on ω. By Corollary 3, we have

$$(Bf(P), Bf(Q); Bf(R), Bf(S)) \stackrel{B}{=} (P, Q; R, S)_\omega \stackrel{C}{=} (Cf(P), Cf(Q); Cf(R), Cf(S)),$$

so $B, C, f(P), f(Q), f(R), f(S)$ are on a conic. If $f(\omega)$ passed through A, then ω would need to be tangent to BC at either B or C, which is impossible.

Note that if $f(\omega) = \omega$ then f defines an involution from ω to itself, and so f must fix exactly two points of ω, which can't both be contained in the same line through B or C. Conversely, suppose for instance that D, E are on ω, and let X be any other point on ω. The conic through $B, C, D, X, f(X)$ is sent to itself, so it must contain E. Thus $f(X)$ must be on ω. ☐

2.1.1 Aside: some basic intersection theory

We recall (without proof) a famous theorem of Bézout.

Theorem 23 (Bézout). *If Ω, ω are distinct curves in $\mathbb{P}^2$ defined by irreducible polynomial equations of degrees m, n, respectively, then the number of intersection points between Ω and ω is exactly mn, if you count points "with multiplicity" and remember to include imaginary points and points at infinity.*

In particular, any two curves in $\mathbb{P}^2$ meet in at least one point. $\mathrm{Bl}_{ABC}\mathbb{P}^2$ doesn't have this property: for instance, the line AB doesn't intersect either of the lines e_C, BC in $\mathrm{Bl}_{ABC}\mathbb{P}^2$. Luckily, it's easy to modify Bézout's theorem to make it work for $\mathrm{Bl}_{ABC}\mathbb{P}^2$.

Definition 19. If ω is a curve in $\mathrm{Bl}_{ABC}\mathbb{P}^2$ defined by an irreducible polynomial equation of degree m, which passes through A, B, C with multiplicities a, b, c, respectively, we say that ω is a *curve of type* $(m, -a, -b, -c)$. If $\omega = e_A$, we say that ω is a curve of type $(0, 1, 0, 0)$, and similarly e_B has type $(0, 0, 1, 0)$, e_C has type $(0, 0, 0, 1)$.

Theorem 24. *If Ω, ω are distinct irreducible algebraic curves in $\mathrm{Bl}_{ABC}\mathbb{P}^2$ of types $(m, p, q, r), (n, x, y, z)$, then the number of intersection points between Ω and ω in $\mathrm{Bl}_{ABC}\mathbb{P}^2$ is exactly $mn - px - qy - rz$, if you count points "with multiplicity" and remember to include imaginary points and points at infinity.*

Definition 20. If ω has type (m, p, q, r), then the *self-intersection number* of ω is defined to be $m^2 - p^2 - q^2 - r^2$.

Proposition 9. *If ω has type (m, p, q, r) then $f_{ABCD}(\omega)$ has type $(2m+p+q+r, -m-q-r, -m-p-r, -m-p-q)$.*

Exercise 43. (a) Prove Proposition 9.

(b) Using Proposition 9 and Theorem 24, check that the number of intersection points between ω and Ω is the same as the number of intersection points between $f_{ABCD}(\omega)$ and $f_{ABCD}(\Omega)$. In particular, the self-intersection number of ω is the same as the self-intersection number of $f_{ABCD}(\omega)$.

(c) Use Proposition 9 to give another proof of Theorem 21.

(d) Find all curves in $\mathrm{Bl}_{ABC}\mathbb{P}^2$ which have self-intersection number at most 0.

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