Chapter 12

The cross ratio

Math 4520, Spring 2015

We have studied the collineations of a projective plane, the automorphisms of the underlying field, the linear functions of Affine geometry, etc. We have been led to these ideas by various problems at hand, but let us step back and take a look at one important point of view of the big picture.

12.1 Klein's Erlanger program

In 1872, Felix Klein, one of the leading mathematicians and geometers of his day, in the city of Erlanger, took the following point of view as to what the role of geometry was in mathematics. This is from his "Recent trends in geometric research."

> Let there be given a manifold and in it a group of transformations; it is our task to investigate those properties of a figure belonging to the manifold that are not changed by the transformation of the group.

So our purpose is clear. Choose the group of transformations that you are interested in, and then hunt for the "invariants" that are relevant.

This search for invariants has proved very fruitful and useful since the time of Klein for many areas of mathematics, not just classical geometry. In some case the invariants have turned out to be simple polynomials or rational functions, such as the case with the cross ratio. In other cases the invariants were groups themselves, such as homology groups in the case of algebraic topology.

12.2 The projective line

In Chapter 11 we saw that the collineations of a projective plane come in two "species," projectivities and field automorphisms. We concentrate on the projectivities. We recapitulate here the results of a series of exercises in Chapter 11. Recall that a projectivity is defined as the composition of projections of the projective plane in some projective 3-space, say. Let Π be a projective plane over some field.

Theorem 12.2.1. *A function from* Π *to itself is a projectivity if and only if it can be described by a non-singular* 3*-by-*3 *matrix in homogeneous coordinates as in Chapter 11.*

So the group that we should use for Klein's program is clear. We should use the projectivities of the projective plane. The connection with linear algebra is clear and direct, and we can use the well-oiled machinery that is easily available. We look for invariants of projectivities.

But before we plunge into that problem, let us step back a bit and see if we can simplify the problem a bit. We have agreed that the group of transformations should be the projectivities, but what if we just look at a single line in the projective plane? If we start over again and look for line preserving functions of a single line, we have not learned our lesson. There is only one line to preserve, and our geometric group is just all functions without regard to any of the geometry outside the line. The proper group to take is the group of functions of the line to itself that extend to a projectivity of the whole projective plane. Alternatively, we could take our group to be those functions, which we call *projectivities*, of the line itself that can be obtained as a composition of projections, each between two lines in the projective plane. See Figure 13.1

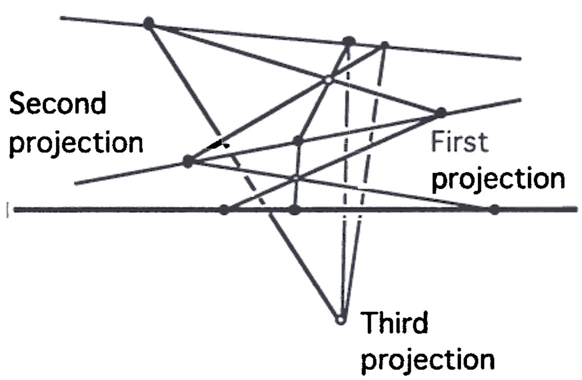

Figure 12.1

Note that this definition for a projectivity of a line is analogous to the definition for a projectivity of a projective plane.

12.3 The extended Affine line and Moebius functions

Regard our projective plane over a field as the extended Affine plane. Look at one line in that plane, say the line $y = 1$. Each pont on that line $\begin{pmatrix} x \\ 1 \end{pmatrix}$ is naturally identified with a field element x. The associated projective line has only one additional point, a point at infinity. Similar to what we did in the case of the whole projective plane, we can define homogeneous coordinates for this projective line, using equivalence classes in the 2-dimensional Affine plane (used as a vector space).

We define a function on this projective line that "comes from" a 2-by-2 matrix A, similar to the function that we defined using a 3 by 3 matrix for the projective plane itself. Let $\begin{pmatrix} x \\ y \end{pmatrix}$

be a vector in the Affine plane. Define a function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is clear that f defines a function on the equivalence classes in the homogeneous coordinate model for the projective line. We next project the functional values back into the original line $y = 1$. See Figure 13.2.

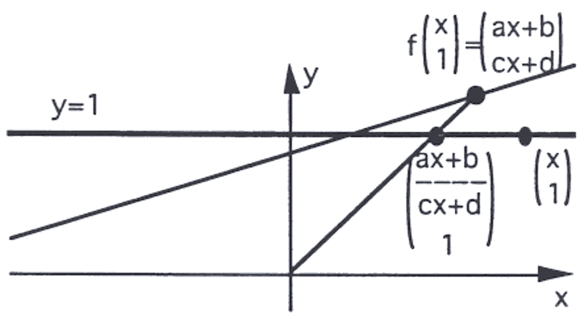

Figure 12.2

So in the extended Affine line itself we have the following function:

$$f(x) = \frac{ax+b}{cx+d},$$

where we interpret the points that come from and go to the point at infinity in the way determined as before. Such a function is called a *Moebius* function.

If we have some other line **l** besides the line $y = 1$, let g be a projectivity of the extended plane that takes **l** onto the line $y = 1$. Then $g^{-1}fg$ defines a Moebius function on the line **l**. In other words a Moebius function is just the function coming from a matrix, but put in Affine coordinates.

Recall the exercises in Chapter 11. There it was shown that a projectivity on the plane or line was equivalent to the function coming from a matrix in homogeneous coordinates. This allows us to pass back and forth between the geometric projectivities and the more analytic Moebius functions. We summarize things in the following Theorem.

Theorem 12.3.1. *Let f be a one-to-one onto function from the points of a line* $\mathbf{l}$ *to itself in a projective plane over a field. Then the following are equivalent:*

(i) The function f is a projectivity of $\mathbf{l}$ *to itself.*

(ii) The function f extends to a projectivity of Π *to itself.*

(iii) The function f is a Moebius function on $\mathbf{l}$ *in the sense above.*

Note that condition (ii) says that any function g can be used to decide whether the function on $\mathbf{l}$ is a Moebius function. If $g^{-1}fg$ is a Moebius function for one projectivity g then it is a Moebius function for all projectivities g.

12.4 Three-point homogeneity

Recall the proof in Chapter 11 that any 4 points, no 3 collinear, can be brought onto any other 4 points, no 3 collinear, by a projectivity of the plane. The proof works in any projective space of any dimension over any field. In dimension d, it is $d + 2$ points. In particular it works for $d = 1$, which we state as follows:

Theorem 12.4.1. *Given any two sets of three labeled distinct points in a line* $\mathbf{l}$ *in a projective plane, there is a projectivity of* $\mathbf{l}$ *that takes the one labeled set onto the other.*

This is essentially Exercise 5 in Chapter 11. Note that, for what it is worth, the projective plane does not have to come from a field, then the proof Chapter 11 does not apply. We call the property of Theorem 13.4.1 *three-point homogeneity* and of course the property of four-point homogeneity in Chapter 11.

12.5 The cross ratio

Finally we look for invariants associated to the group of projectivities of a line $\mathbf{l}$ in a projective plane Π over a field $\mathbf{F}$. Ideally we want our first invariants to be as simple as possible, involving as few points as possible. However, we know, by three-point homogeneity on the line, that if the invariant involves three or fewer points, then it does not depend on which three points, and the invariant will be trivially a constant. So we look for invariants of four points on a line.

In order to calculate the invariant, we need to have a field structure on $\mathbf{l}$. The invariant will be an element of $\mathbf{F} \cup \{\infty\}$. In other words, the invariant might be "infinite," but otherwise, it is in the field $\mathbf{F}$. Let g be any projectivity of Π that takes $\mathbf{l}$ to the line $y = 1$, say. So a point $\mathbf{p}$ on $\mathbf{l}$, associated with $g(\mathbf{p})$, is now regarded as an element of $\mathbf{F} \cup \{\infty\}$. With this in mind we consider our four distinct labeled points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ on $\mathbf{l}$. We define the *cross ratio* of these points as

$$r = \left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{\mathbf{p}_1 - \mathbf{p}_4}\right)\left(\frac{\mathbf{p}_3 - \mathbf{p}_4}{\mathbf{p}_3 - \mathbf{p}_2}\right)$$

We adopt the same conventions for r as we did for the Moebius functions about points that go to and come from infinity. There is never any ambiguity when the four points are distinct or even when exactly one pair is the same. We make a few observations about r.

1. Each $\mathbf{p}_i$ appears once in the numerator and once in the denominator, each time with the same sign. In fact, multiplying each $\mathbf{p}_i$ by the same constant does not change the value of r.

2. The cross ratio is a function of the differences of pairs of points. Thus adding the same constant to each $\mathbf{p}_i$ does not change the value of r.

3. Replacing each $\mathbf{p}_i$ by $1/\mathbf{p}_i$ does not change the value of r.

4. Regard $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, as constants. Then r is a Moebius function of $\mathbf{p}_4$.

In the classical case of the real projective plane a slightly different point of view is taken. Instead of taking any projectivity g from the line $\mathbf{l}$ to the standard line with the field structure, usually $\mathbf{l}$ is considered as "oriented" and the algebraic differences in the definition of r are taken as oriented distances between the appropriate points. The only ambiguity in the definition of r is which orientation to choose for $\mathbf{l}$, and clearly this does not change the value of r. This is clearly equivalent to choosing g to be some rigid congruence that takes $\mathbf{l}$ to the standard line with its field structure. However, a metric distance approach seems out of place here since it is not the metric geometry that we are studying now. Also for fields other than the real field, we may not have any convenient metric to measure distances anyway, if that is the way we insist on defining r.

12.6 The invariance of the cross ratio

We need to show that the cross ratio is invariant when the four points defining it are mapped to four others by a Moebius function.

Lemma 12.6.1. *Let* $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ *be four distinct points in* $\mathbf{F} \cup \{\infty\}$. *Let* f *be a Moebius function on* $\mathbf{F} \cup \{\infty\}$, *and let* $\mathbf{q}_i = f(\mathbf{p}_i)$, $i = 1, 2, 3, 4$. *Then the cross ratio for* $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ *is the same as for* $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$.

Proof 1. Any Moebius function can be written as a composition of three types of functions, those that add a constant, those that multiply a constant, and those that take the multiplicative inverse. Explicitly,

$$f(x) = \frac{ax+b}{cx+d} = \frac{\frac{a}{c}(cx+d) + b - \frac{ad}{c}}{cx+d} = \frac{a}{c} + \left(b - \frac{ad}{c}\right)\frac{1}{cx+d}\,,$$

which is seen to be a composition involving two multiplications, two additions and one inverse. By observations 1, 2, 3 in Section 13.5, we see that each of the three types of functions preserve the cross ratio, so any Moebius function does. □

Proof 2. Here we regard the function f as coming from a linear function in homogeneous coordinates and use basic properties of the determinate. Regard the four points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ as being on the line $y = 1$. Then each $\mathbf{q}_i$ is the projection of $A\mathbf{p}_i$ onto the line $y = 1$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let

$$A\mathbf{p}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix},$$

for $i = 1, 2, 3, 4$. So then

$$f(\mathbf{p}_i) = \frac{x_i}{y_i} = \mathbf{q}_i$$

regarded as an element of $\mathbf{F}$. Then

$$\mathbf{q}_i - \mathbf{q}_j = \frac{x_i}{y_i} - \frac{x_j}{y_j} = \frac{x_i y_j - x_j y_i}{y_i y_j} = \frac{1}{y_i y_j} \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} = \frac{1}{y_i y_j} \det(A\mathbf{p}_i, A\mathbf{p}_j),$$

where 'det' represents the determinate function. So the cross ratio of the $\mathbf{q}_i$'s is

$$r = \frac{\det(A\mathbf{p}_1, A\mathbf{p}_2)\det(A\mathbf{p}_3, A\mathbf{p}_4)}{\det(A\mathbf{p}_1, A\mathbf{p}_4)\det(A\mathbf{p}_3, A\mathbf{p}_2)}.$$

But a basic property of determinants implies

$$\det(A\mathbf{p}_i, A\mathbf{p}_j) = \det A \det(\mathbf{p}_i, \mathbf{p}_j).$$

Thus the factors $\det A \neq 0$ cancel, and r is the same for both the $\mathbf{p}_j$'s and the $\mathbf{q}_j$'s. □

We can now observe by Theorem 13.3.1 that not only is the cross ratio r invariant under Moebius functions and projectivities of projective lines and projectivities of the projective plane, but its definition is clearly independent of the choice of the projectivity g used to define the field structure in Section 13.5. We finally know that r is well-defined.

12.7 Harmonic points

Consider the special case when $\mathbf{p}_2 = 0$, $\mathbf{p}_3 = 1$, $\mathbf{p}_4 = \infty$. Then $r = \mathbf{p}_1$. So these canonical choices of the points help us to understand the role that the cross ratio plays with respect to the four points used to define it. By three-point homogeneity, we can always reposition three of our four points to be in the special position we choose above.

When our four labeled points have cross ratio -1, we say that they are *harmonic points.* Consider the configuration in Figure 13.3.

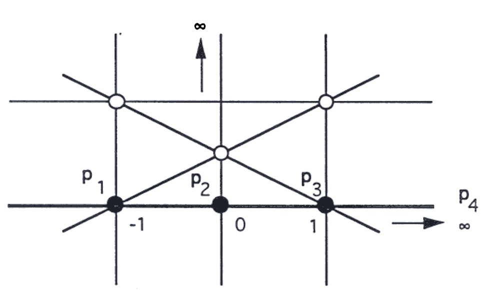

Figure 12.3

Clearly if our four points have cross ratio -1, then such a configuration as in Figure 13.3 exists. Use a projectivity g to take three of the four points to 0, 1, ∞, and then the fourth is forced to go to -1, since their cross ratio is -1. Then map the configuration back by the inverse of g.

On the other hand, it is clear from the configuration of Figure 13.3 that if $\mathbf{p}_1$, $\mathbf{p}_3$, $\mathbf{p}_4$ are at -1, 1, and ∞, respectively, then $\mathbf{p}_2$ is 0. So by a similar argument, if such a configuration exists, then the points have cross ratio -1. An example is shown in Figure 13.4, where the construction is done twice, once on each side of the line.

It is clear that if one construction says that the points are harmonic, then the other must also. Note that any one of the white points in the construction can be chosen arbitrarily not on the line of the points.

Note also the similarities of Figure 13.4 and the constructions of grids in earlier chapters.

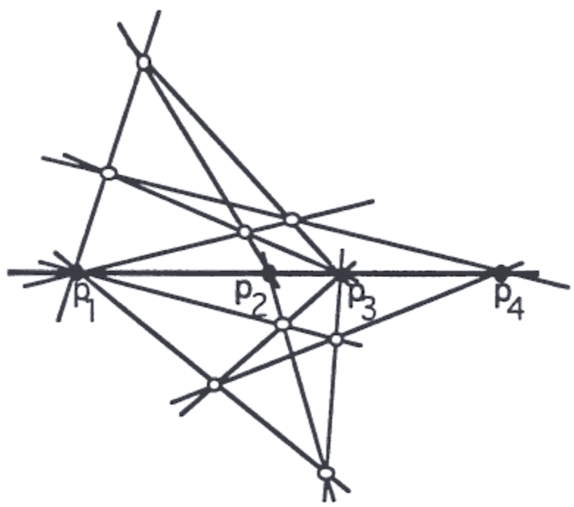

Figure 12.4

12.8 Exercises:

1. There are 24 permutations of 4 distinct points.

   (a) For 4 distinct points on a line, how many distinct values are there for the cross ratio of all permutations of the points?

   (b) If one of the cross ratios is t, find the others as a function of t.

   (c) If the four points are harmonic, how many distinct cross ratios are there, and what are they?

2. Suppose that points $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ are given on a line. Find a construction as in Section 13.7 for finding $\mathbf{p}_1$ such that the cross ratio is 3.

3. Consider the Homogeneous Coordinate Model 2 for the projective plane over a field.

   (a) Make a proper definition of what is meant by the cross ratio of 4 labeled lines that are incident to a single point.

   (b) Show that the cross ratio that you defined in Part (3a) is the same as the cross ratio of the 4 points on a line as in Figure 13.5.

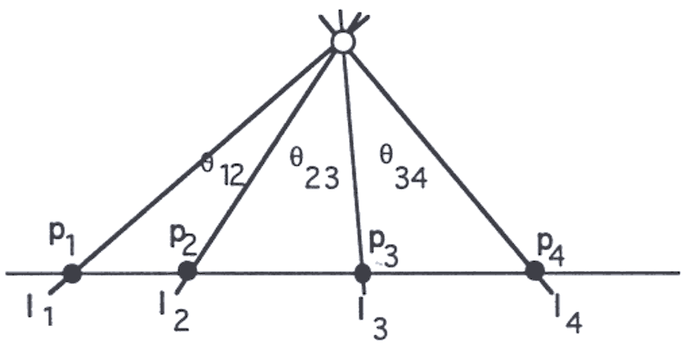

Figure 12.5

   (c) Find an expression for the cross ratio of the 4 lines above in terms of the 3 angles θ_{12}, θ_{23}, θ_{34}.

4. Suppose that we have 4 lines all incident to a point **p** on a circle C as in Figure 13.6.

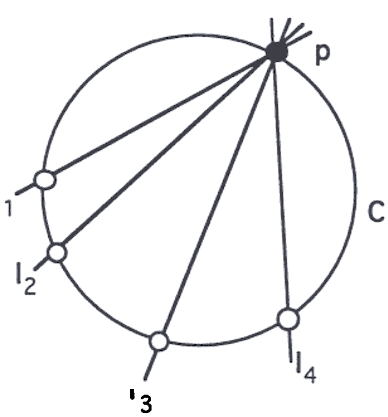

Figure 12.6

Show that as **p** varies along C with the white points fixed, then the cross ratio of the 4 labeled lines is constant.

5. Find the function that plays the role of the Moebius function when the underlying field is not commutative. Is such a Moebius function determined uniquely by its values on 3 distinct points?

6. In the real projective plane show that the cross ratio of 4 points $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ is negative if and only if $\mathbf{p}_1$ and $\mathbf{p}_3$ are separated by $\mathbf{p}_2$ and $\mathbf{p}_4$.