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ACCURACY, RELIABILITY AND STATISTICS IN CLOSE-RANGE PHOTOGRAMMETRY

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A. Introduction

Analytical close-range systems for point determination are characterized by typical features, which differ to some extent from the conditions found in aerial triangulation. The main problem consists in the great variety of quite different conditions with respect to image geometry, interior orientation, exterior orientation, intersection conditions, control distributions and systematic errors. In the past this led to many different solution procedures, often reconciled to the presumptions of a special project.

While in aerial triangulation the handling of large linear systems mostly complicates the application of highly sophisticated mathematical and statistical techniques this is not true in close-range photogrammetry, where the systems generally are smaller and thus far more convenient to work with in computer programs. So the creation and application of general, highly developed methods is regarded as a basic requirement in close-range photogrammetry, especially if the results have to be of high accuracy and reliability. Thus rigorous procedures of network design determination (*Grafarend /8/, Schmitt /20/*) and a-posteriori variance estimation (*Ebner /5/, et.al.*) nowadays mainly used in geodesy should find more attention in photogrammetry.

Even relatively simple a-priori accuracy studies on the applied bundle system are not very popular, though in some non-transparent situations absolutely necessary. Moreover the reliability of the systems should be studied with more attention. Based on *Baarda's* reliability theory /2/, /3/ this topic is strongly connected with the problem of model errors and mainly used in connection with gross error detection. A system may be called "reliable" if gross errors of a certain size can be detected with a certain statistical security. It is important to notice that a system or parts of it may be accurate without being reliable at all. It is one objective of this paper to mark off the terms "accuracy" and "reliability" with the aid of some simple examples.

A highly developed bundle model must include the self-calibration technique. A comprehensive compensation of systematic image errors requires a general and flexible additional parameter set. Thus the concept of bivariate orthogonal polynomials is recommended and a suitable set for a 5 x 5 image point distribution is presented.

In order to perform a widely objective analysis of bundle adjustment results reference is made to general statistical methods and some test criterions are derived to test hypotheses often appearing in point determination problems.

B. The mathematical model of self-calibrating bundle adjustment and the problem of systematic errors.

In close-range photogrammetry it is generally noticed that with an increasing number of different problems the number of mathematical models increases. This leads to a certain confusion and reduces the compatibility and applicability of existing computer programs.

So a standardization of models seems to be necessary, which may be based on a general and flexible self-calibrating bundle adjustment having proved its efficiency in aerial triangulation. For the further analytical treatment reference is made to the model of the existing bundle program MBOP (Munich Bundle Orientation with Additional Parameters), originally developed for aerial triangulation problems:

$$
\begin{aligned}
-e^B &= A_1 dx + A_2 dx^P + A_3 dt + A_4 dz - l^B \;; && P_B\ (=I) \\
-e^Z &= \qquad\qquad\qquad\qquad\qquad I\ dz - l^Z \;; && P_Z \\
-e^P &= \qquad\quad I\ dx^P \qquad\qquad\qquad - 0 \;; && P_P
\end{aligned}
\tag{1}
$$

e^B, e^Z, e^P = Vectors of true errors of image coordinates, additional parameters, control point coordinates

dx, dx^P, dt, dz = Correction vectors of point coordinates, control point coordinates, elements of exterior orientation, additional parameters

A_1, A_2, A_3, A_4 = Corresponding coefficient matrices

$l^B, l^Z, 0$ = Vectors of observations of image coordinates, additional parameters, control point coordinates

I = Identity matrix

Collecting the matrices and vectors of (1) in the following manner

$$
\underset{n,u}{A} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{pmatrix} , \qquad \underset{n,n}{P} = \begin{pmatrix} P_B & 0 & 0 \\ 0 & P_Z & 0 \\ 0 & 0 & P_P \end{pmatrix} ,
$$

$$
\underset{1,u}{x^T} = (dx^T,\ dx^{P^T},\ dt^T,\ dz^T) \;, \quad \underset{1,n}{l^T} = (l^{B^T},\ l^{Z^T},\ 0^T) \;,
$$

$$
\underset{1,n}{e^T} = (e^{B^T},\ e^{Z^T},\ e^{P^T})
$$

the system (1) formally results in

$$
-e = Ax - l \;; \quad P \tag{2}
$$

and by using the operator of expectation E and the operator D which leads to the variances and covariances we get in statistical notation

$$
\begin{aligned}
&AE(x) = E(l) \;, \\
&E(x) = \mu_x \;, \quad E(l) = \mu_l \;, \quad E(e) = 0 \;, \\
&D(e) = \sigma_o^2 P^{-1} \;, \qquad (\sigma_o = \text{standard deviation of unit weight to be estimated})
\end{aligned}
\tag{3}
$$

Thereby the correction vectors of the ground point coordinates dx and of the elements of exterior orientation dt are always introduced as free unknowns, what generally leads to mixed models.
System (3) is called a Generalized Gauß-Markov Model (*Koch /12/*). For known expected values μ_x and for $P_Z \neq 0$ it may be interpreted as (mixed) Regression Model, for $P_Z \neq 0$, $E(l^Z) = 0$ as (mixed) Collocation Model (*Ebner /6/*). If the geodetic

control point coordinates are introduced as free of errors (diag $(P_P) \to \infty$), like it is often required in practice, and if the additional parameters are treated as free unknowns ($P_Z = 0$), the ordinary Gauß-Markov Model, i.e. the usual case of Adjustment of Observations is obtained.
The matrix P_P allows to introduce a-priori known accuracy standards of the control point coordinates; the same applies to the matrix P_Z and the additional parameters. Using $P_P = 0$ the system becomes singular.
Under certain circumstances and with certain parameter sets the treatment of systematic errors as free unknowns ($P_Z = 0$) may lead to remarkable deteriorations of the condition of the system of normal equations. Therefore it is expedient to introduce $P_Z \neq 0$, thus protecting to some extent against overparametrization.
With $P_P \neq 0$ we obtain a weighted minimal fitting (with respect to a 3-dimensional similarity transformation) of the photogrammetric points onto the control points.

With the set of additional parameters from (1) we are fully flexible, there is no restriction concerning their number and type. This enables us to introduce block-invariant parameters, just as parameters which belong to a single strip, to a certain group of images or even to a single image.
In close-range applications this concept has to be extended by observation equations for additional measurement elements as angles, distances and exterior orientation parameters and by some special conditions as straight line conditions, surface conditions, angle conditions and so forth. For the purpose of the following studies these equations can be omitted without loss of generality.
The same applies to other image geometries which may replace the perspective relations.

For the determinability of systematic errors and the related eigenvector problem see *Grün /11/*.
The problem of a suitable choice of an additional parameter set was recently discussed in *Grün /10/*. Thereby the functional, numerical and statistical advantages of bivariate orthogonal parameters, introduced by *Ebner /6/*, have been emphasized.

A strategy in additional parameter construction, i.e. the choice of the design matrix A_4 (see (1)) must consider two basic requirements:

- The estimation of x and σ_o^2 in (3) has to be unbiased, i.e. the systematic deformations have to be modeled as well as possible
- The variances of the additional parameters (and of the other parameters of system (1)) should be as small as possible.

To the first point experiences, gained over many years, have shown that polynomials are a proper device in modelling systematic image errors. Thereby the favourable property of bivariate polynomials consists in its capability to compensate the total systematic effect at all points of a presumed regular image point screen (*Ebner /6/*).
The second requirement, which includes equally the demand for minimal covariances, leads to the condition

$$Q = (A^T P A)^{-1} \Longrightarrow \text{Min} \quad , \tag{4}$$

or if only the additional parameters are regarded

$$Q_{ZZ} = (A_4^T P_B A_4)^{-1} \Longrightarrow \text{Min} \quad . \tag{5}$$

As it is proved in *Rao /18/*, the minimum of Q_{ZZ} is obtained when

$$A_{4i}^T P_B A_{4j} = 0 \quad , \qquad \begin{array}{l} i = 1,\ldots,z \\ j = 1,\ldots,i-1,\ i+1,\ldots,z \\ z = \text{number of add. parameters} \end{array} \tag{6}$$

i.e. for a diagonal weight matrix P_B the optimal choice of A_4 is obtained, if the columns of A_4 are orthogonal, not necessarily orthonormal. (Hence orthogonality is defined as : $A^T PA = D$, $D =$ diagonal matrix).

Although the concept of orthogonality is strongly valid only in the case of extremely regular network arrangements, orthogonal sets provide in practice for the best possible independence.

It is clear that in aerial triangulation systems orthogonality is more likely than in close-range systems.

For a 3 x 3 image point distribution the corresponding orthogonal additional parameter set was derived by *Ebner /6/*. Recently the author developed an orthogonal set with respect to a 5 x 5 image point distribution.

With $k = x^2 - \frac{b^2}{2}$, $l = y^2 - \frac{b^2}{2}$, $p = x^2 - \frac{17}{20} b^2$, $q = y^2 - \frac{17}{20} b^2$

$$r = x^2(x^2 - \frac{31}{28} b^2) + \frac{9}{70} b^4 \quad , \qquad s = y^2(y^2 - \frac{31}{28} b^2) + \frac{9}{70} b^4$$

we obtain

$$\begin{array}{l} \Delta x = a_{12}x + a_{21}y + a_{22}xy + a_{31}l + 0 - b_{22}\frac{10}{7}k + \\ \Delta y = -a_{12}y + a_{21}x - a_{22}\frac{10}{7}l + 0 + b_{13}k + b_{22}xy + \end{array} \tag{7}$$

$$\begin{array}{l}
(\Delta x..) + a_{14}xp + a_{23}yk + a_{32}xl + a_{41}yq + 0 + 0 + 0 + 0 + \\
(\Delta y..) + 0 + 0 + 0 + 0 + b_{14}xp + b_{23}yk + b_{32}xl + b_{41}yq + \\[1ex]
(\Delta x..) + a_{15}r + a_{24}xyp + a_{33}kl + a_{42}xyq + a_{51}s + 0 + 0 + 0 + 0 + 0 + \\
(\Delta y..) + 0 + 0 + 0 + 0 + 0 + b_{15}r + b_{24}xyp + b_{33}kl + b_{42}xyq + b_{51}s + \\[1ex]
(\Delta x..) + a_{25}yr + a_{34}xlp + a_{43}ykq + a_{52}xs + 0 + 0 + 0 + 0 + \\
(\Delta y..) + 0 + 0 + 0 + 0 + b_{25}yr + b_{34}xlp + b_{43}ykq + b_{52}xs + \\[1ex]
(\Delta x..) + a_{35}lr + a_{44}xypq + a_{53}ks + 0 + 0 + 0 + \\
(\Delta y..) + 0 + 0 + 0 + b_{35}lr + b_{44}xypq + b_{53}ks + \\[1ex]
(\Delta x..) + a_{45}yqr + a_{54}xps + 0 + 0 + a_{55}rs + 0 \quad ; \\
(\Delta y..) + 0 + 0 + b_{45}yqr + b_{54}xps + 0 + b_{55}rs \quad ;
\end{array}$$

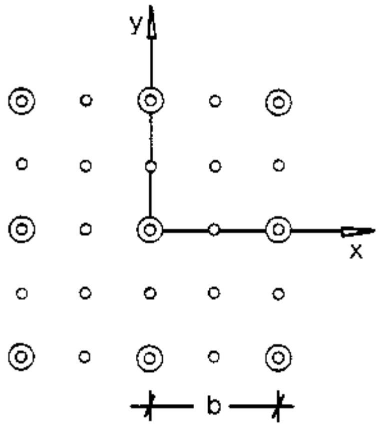

Figure 1: Regular image point screens
◎... 3x3 distribution
o... 5x5 distribution

For vertical photography and flat object this set is outlined to be even independent on the exterior orientation elements. Under very general conditions (large rotation elements, space character of the object) the corresponding sets can be extended by some of the 6 coefficients which are highly correlating with the 6 exterior orientation elements in the case of vertical photography and flat object to obtain a complete bivariate set (here the parameters of interior orientation are also included). With the creation of the complete set one has to pay attention to the sequence of the rotation elements. For example, if κ is chosen dependent on ϕ and ω then one of the two linear terms which lead to image torsions has to be rejected a-priori.
Because of the possibly high correlations the extended sets have to be applied very carefully. To protect against overparametrization, i.e. to avoid the introduction of remarkable instabilities into the linear system (1) sophisticated statistical test methods must be used for additional parameter significance testing in any case (see *Grün /10/, /11/* and also section D1 of this paper). One has to note that the 1-dimensional Student test leads to wrong conclusions concerning the acceptance or rejection of individual additional parameters if dangerous correlations do occur. Thus a more comprehensive treatment of statistical hypotheses testing methods becomes necessary.

C. Accuracy and reliability of close-range bundle systems

Hitherto in photogrammetric close-range systems the accuracy was almost exclusively in the center of interest. In this connection sometimes a lot of work was invested to derive "accuracy predictors" which should represent a kind of accuracy models for special, often appearing network arrangements. Nevertheless, among all accuracy predictors the inverse of the normal equations of a bundle system is still the best one. It accommodates every changing network configuration and every model variation. So the design of a project should be connected with the determination of the corresponding inverse for sophisticated accuracy studies. This becomes important more than ever, since the introduction of additional parameter sets complicates an empirical accuracy appraisal. Besides the accuracy of a system its reliability should become a main objective of future studies.
The reliability of a model describes its sensitiveness with respect to model errors. If a highly developed self-calibrating bundle model is used we may restrict our considerations on the problem of gross errors.

So by definition the term "reliability" should describe the system's qualities for gross error detection.
It was *Baarda /2/, /3/* who developed a rather complete reliability theory. His approach enables the treatment of the gross error detection problem on a statistical basis. *Baarda's* theory may equally be applied to theoretical reliability studies of bundle adjustment systems as to the detection of gross errors in practical projects. For detailled information about the whole theory see */2/, /3/*. In the following only a short extract is presented, as far as it is necessary to understand the practical computations presented in this paper.
Starting from system (1) with the statistical model

$$\begin{aligned} -e &= Ax - l \quad ; \quad P \\ D(e) &= D(l) = \sigma_o^2 P^{-1} \quad , \quad E(e) = 0 \end{aligned} \tag{8}$$

we get consistent, sufficient and minimum variance unbiased estimators for x and σ_o^2 by the least squares estimators (with redundancy $r = n-u$)

$$\hat{x} = (A^T PA)^{-1} A^T Pl = Q_{xx} A^T Pl \quad , \tag{9a}$$

$$\hat{\sigma}_o^2 = \frac{1}{r} (A\hat{x}-l)^T P(A\hat{x}-l) = \frac{v^T Pv}{r} \quad , \tag{9b}$$

and for the residuals we obtain

$$v = A\hat{x} - l = -(I - AQ_{xx}A^T P)l \quad , \tag{10}$$

and with

$$Q_{vv} = P^{-1} - AQ_{xx}A^T \tag{11}$$

we get

$$v = -Q_{vv} Pl \quad . \tag{12}$$

Regarding model (8) as null-hypothesis H_o with $E(\hat{\sigma}_o^2/H_o) = \sigma_o^2$, then if H_o is true the test criterion

$$\Theta = \frac{\hat{\sigma}_o^2}{\sigma_o^2} \tag{13}$$

is distributed as the central F-distribution $F(r,\infty)$. Thus Θ is used as a global test criterion to test the presumed multi-dimensional normal distribution of l ($l \sim N(Ax, \sigma_o^2 P^{-1})$).
If H_o is rejected one has to set up one or more alternative hypotheses H_A.
Since the systematic errors are widely compensated by our self-calibrating model we may confine ourselves on the investigation of gross errors. Then a set of p alternative hypotheses H_{A_p} can be formulated as

$$H_{A_p}: \quad \nabla l_p = \sigma_o c_p \nabla_p \quad , \qquad \begin{aligned} p &= 1,\dots,p \text{ alternative hypotheses} \\ c_p &= \text{vector representing the proportions of the gross errors} \\ \nabla_p &= \text{constant} \end{aligned} \tag{14}$$

If H_{A_p} is accepted, then Θ is distributed as the noncentral F-distribution $F(r,\infty,\lambda_p)$ with the noncentrality parameter λ_p.

H_{A_p} leads to

$$E(\hat{\sigma}_o^2 / H_{A_p}) = E(\hat{\sigma}_o^2 / H_o) + \nabla\sigma_{op}^2 \quad , \qquad (15a)$$

$$\nabla\sigma_{op}^2 = \frac{\lambda p}{r}\sigma_o^2 \quad . \qquad (15b)$$

λ_p depends on the type I error size α, on the type II error size $\beta(\lambda_p)$ ($\beta(\lambda_p)$ = 1-P(II), P(II) = probability of accepting a false hypothesis) and on the degrees of freedom r, ∞. For a given $\alpha,\beta(\lambda_p)$, r we can find λ_p by using *Baarda's* nomograms in */3/*, pp. 21-23.

If H_{A_p} is true, then the estimators $\hat{x}$ and $\hat{\sigma}_o^2$ are no more unbiased. If the effect of a gross error vector ∇l_p in the observations upon the residual vector v has to be computed then we obtain

$$E(v/H_{A_p}) = E(v/H_o) + \nabla v_p \quad ; \qquad (16)$$

$$\nabla v_p = - Q_{vv}P \nabla l_p \quad . \qquad (17)$$

Although in practical projects we normally don't know anything about the vector c_p - it indicates the relationship of the gross errors - equation (17) enables us to study the effect of an a-priori supposed gross error vector ∇l_p upon the residual vector v. From (14), (15), (17) and from the idempotency of the matrix product $Q_{vv}P$ it follows for small gross errors

$$\lambda_p = N_p \nabla_p^2 \quad , \quad N_p = c_p^T P Q_{vv} P c_p \quad . \qquad (18)$$

The vector ∇l_p contains all assumed gross errors of the pth alternative hypothesis. If we suppose only one gross error in the observation i we get

$$H_{A_1}: \ \nabla l_1^{(i)} = \sigma_o c_1 \nabla_1 \quad , \qquad (19)$$

and with $c_1^T = (0,...0,1,0,...,0)$

$$H_{A_1}: \ \nabla l_1^{(i)} = \sigma_o \nabla_1 \quad . \qquad (20)$$

combining (18) and (20) we obtain

$$H_{A_1}: \ \nabla l_1^{(i)} = \sigma_o \sqrt{\frac{\lambda_1}{N_1}} \quad , \qquad (21)$$

for diagonal P: $N_1 = p_i^2 q_{v_i v_i}$, $q_{v_i v_i}$ = i^{th} diagonal element of Q_{vv}

i.e. $\nabla l_1^{(i)}$ is the minimal gross error which can be detected with the probaility $\beta(\lambda_1)$. Thus for a constant λ_1 for all possible i $\nabla l_1^{(i)}$ represents a suitable individual reliability indicator ("measure for internal reliability" in *Baarda /3/*).

Furthermore, *Baarda /3/* has developed a test criterion for testing the residuals of an adjustment ("data-snooping"). If H_o is rejected, we get

$$w_p = \frac{-c_p^T P v}{\sigma_o \ N_p} \quad . \qquad (22)$$

With one of the possible alternative hypotheses $(c_1,..., c_p)$ = I from (14) and by using a weight matrix of diagonal form we obtain

$$w_i = \frac{-v_i}{\sigma_o \sqrt{q_{v_i v_i}}} \quad . \tag{23}$$

The acceptance interval for w_i is then

$$- F^{1/2}(1-\alpha_o,1,\infty) < w_i < F^{1/2}(1-\alpha_o,1,\infty) \quad , \tag{24}$$

where α_o is the type I error size for the test criterion w_i.

The significance level α_o for the test of individual residuals (23) is dependent on the significance level α of the global test (13). *Baarda* harmonizes α with α_o by putting $\beta(\lambda_p) = \beta(\lambda_1)$, i.e. by using equal type II error sizes for the global test and the individual test. For the relations between α and α_o see the nomograms in */3/*, pp. 21-23. As far as the author could understand this, the harmonizing between α and α_o was based on the assumption that the single events w_i are independent on each other. Hence a most reliable system is required for an effective application of the data-snooping technique; otherwise the danger of false inference is too great. At least we have to exclude those observations from the test procedure - or better: we have to change the network arrangement in a way that those observations don't occur - which are correlating too much with others. This is valid especially for "spur" observations, i.e. for observations which lead to $\sigma_v = 0$ (zero variance problem), since in the extreme case of spur observations the test criterion $w_i = \frac{0}{0}$ is not defined.

In (21) $\nabla l_1^{(i)}$ was denoted as an individual reliability indicator. The amount for the computation of all individual $N_1 = p_i^2 q_{v_i v_i}$ (if P is diagonal and if only one gross error is assumed) may sometimes prevent of individual computations. However we are able to make some statements on the global reliability of a system (see *Förstner /7/, Grün /11/, Pope /17/*).

In *Grün /11/* the average diagonal term of Q_{vv} was proposed to serve as global reliability indicator:

$$RI(T) = \frac{tr(Q_{vv})}{n} = \frac{r}{n} \quad , \tag{25a}$$

$$RI(x') = \frac{2tr(Q_{vv}^{x'})}{n} \quad , \quad RI(y') = \frac{2tr(Q_{vv}^{y'})}{n} \quad . \tag{25b}$$

r = redundancy, n = number of observations

Of course global reliability indicators can be constructed which do more refer to the "external" reliability (*Baarda /3/*), that means to the effect of gross errors upon the final product of the adjustment - the object point coordinates. Those would correspond even more with the following accuracy indicators. But here the term "reliability" should be stronger connected with the detection than with the effect of gross errors.

An analogous global accuracy indicator would be $\frac{tr(Q_{xx})}{3k}$, yet we use the average standard deviations of the point coordinates (with $\sigma_o = 1$):

$$AI(T) = \frac{tr(\sqrt{Q_{xx}})}{3k} \quad , \qquad k = \text{number of points, including the control points} \tag{26a}$$

$$AI(X) = \frac{tr(\sqrt{Q_{xx}^X})}{k} \quad ,$$

$$AI(Y) = \frac{tr(\sqrt{Q^Y_{xx}})}{k} \quad , \qquad (26b)$$

$$AI(Z) = \frac{tr(\sqrt{Q^Z_{xx}})}{k} \quad .$$

In the following a simple practical example is introducted to clarify the above mentioned problems and to show the practical potential of the data-snooping technique.

Although in *Grün /11/* *Pope's* approach of the gross error detection problem was emphasized, in this paper *Baarda's* approach is used because the values of the Tau - distribution (*Pope /17/*) are not available for the author by now.

Another interesting method was proposed by *Benning /4/* which should also receive future attention.

For further demonstration the practical example of Figure 2 is introduced, based on synthetic data.

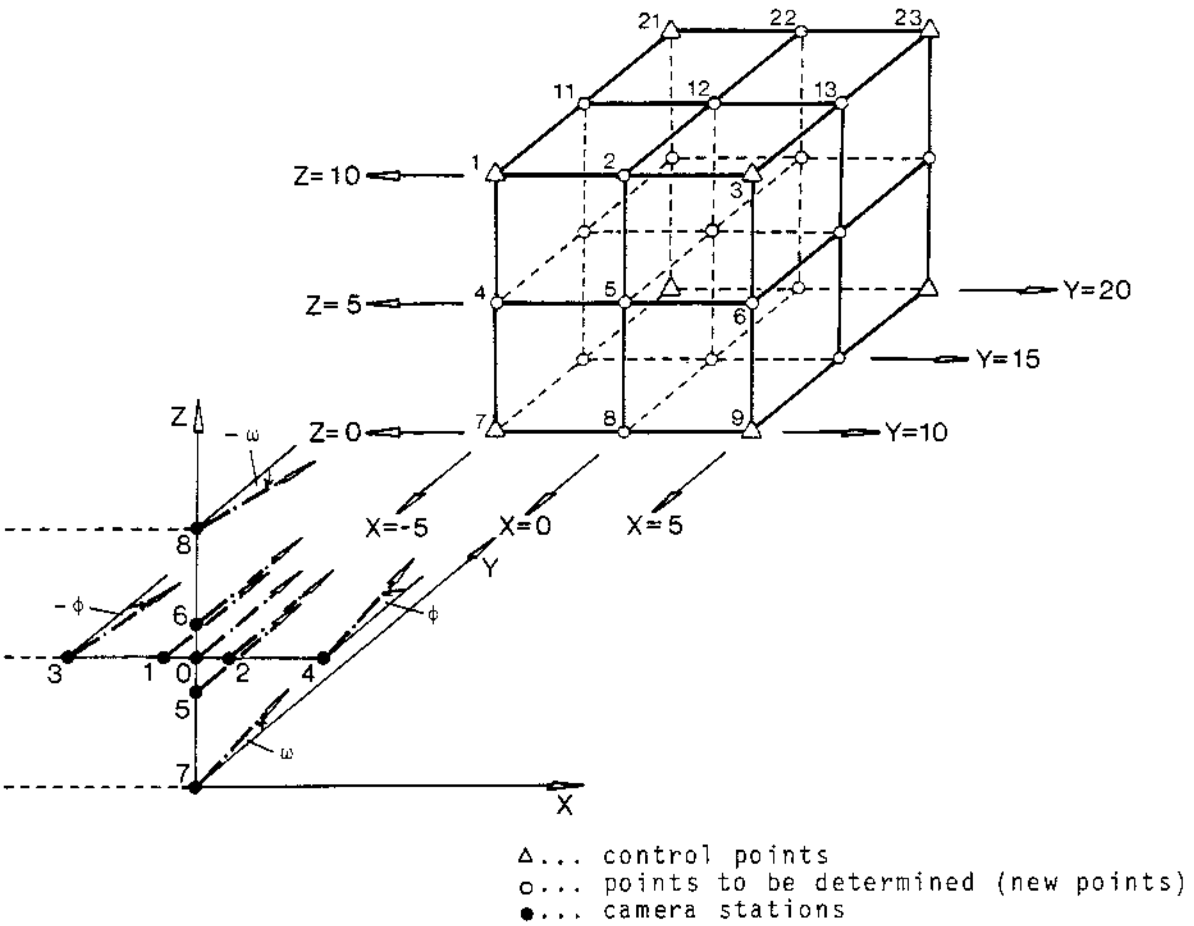

Figure 2: Network arrangement for the demonstration of accuracy and reliability problems

We suppose a cube which contains 27 regularly distributed points; 8 of them serve as control points, the others are new points. The camera stations are denoted by nos. 0, ..., 8.

Table 1 shows the orientation elements of the individual images.

Table 1: Orientation elements of the images of the network arrangement of Figure 2

Images	c [mm]	X_o [m]	Y_o [m]	Z_o [m]	κ [g]	ϕ [g]	ω [g]
0	100	0	0	5	0	0	0
1	100	-1	0	5	0	0	0
2	100	1	0	5	0	0	0
3	100	-5	0	5	0	-20.483	0
4	100	5	0	5	0	20.483	0
5	100	0	0	4	0	0	0
6	100	0	0	6	0	0	0
7	100	0	0	0	0	0	20.483
8	100	0	0	10	0	0	-20.483

While the object points and the control distribution remains unchanged in the following computational versions, the images are grouped as seen from Table 2. For simplification all versions are computed without additional parameters and with fixed control points (diag $(P_p) \rightarrow \infty$).

Table 2: Image arrangements and global accuracy and reliability indicators AI(X), AI(Y), AI(Z), AI(T), RI(x'), RI(y'), RI(T); σ_o = 1 µm

Version	Image arrangement	n	u	r	Accuracy indicators AI(X) [mm]	AI(Y) [mm]	AI(Z) [mm]	AI(T) [mm]	Reliability indic. RI(x')	RI(y')	RI(T)
A	1-2	108	69	39	0.27	1.17	0.27	0.57	0.19	0.53	0.36
B	3-4	108	69	39	0.09	0.29	0.11	0.16	0.20	0.52	0.36
C	3-0-4	162	75	87	0.08	0.28	0.10	0.15	0.43	0.64	0.54
D	1-2-5-6	216	81	135	0.18	0.92	0.18	0.43	0.62	0.62	0.62
E	3-4-7-8	216	81	135	0.07	0.21	0.07	0.12	0.62	0.62	0.62

n = number of observations

u = number of unknowns

r = n-u = redundancy

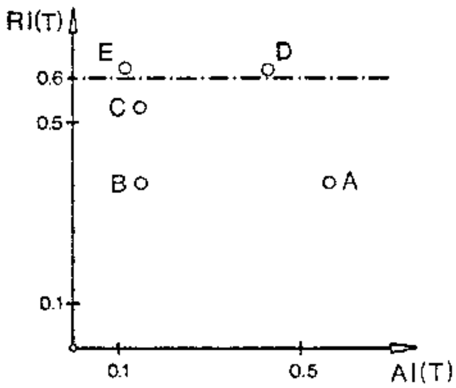

Figure 3: Global reliability and accuracy indicators of the practical examples of Table 2

Table 2 together with Figure 3 show the global reliability and accuracy indicators. Some interesting conclusions can be drawn from these investigations. Comparing the global indicators RI(T) and AI(T) of the individual network versions we notice that good accuracy doesn't correspond necessarily with good reliability. Though version B provides for fairly good accuracy, the reliability is poor. This refers mainly to the reliability of the x'-coordinate observations. Here we have to state that if only two images are available (see also version A) a gross error of those image coordinates which belong to the epipolar plane cannot be detected at all (zero variance problem - "spur" observations).
Of course this is not true for control points, so that RI(x') of version A, which represents the average variance of the x'-residuals (with σ_c = 1 µm) is not equal to zero. Though the x'-coordinates of version B don't belong exactly to the epipolar plane ($\phi_4 = -\phi_3 = 20.4839$), their reliability is still poor (that yields for the non-control points). On the contrary the reliability of the y'-coordinates of versions A,B is sufficient (A:RI(y') = 0.53, B:RI(y') = 0.52), i.e. a gross error in those y'-coordinates has a good chance to be detected (see the subsequent examples of data - snooping); but here another problem reveals: the problem of gross error localization.
Example C (3 images) provides for better reliability, especially in the x'-coordinates (RI(x') = 0.43). In this case we are able to detect a gross error even in the x'-coordinates of non-control points.
The examples D and E show an equally good reliability behaviour in x'- and y'- coordinates, what is due to the symmetric network arrangement, while the accuracy of version E is far more better.
Summarizing the experiences gained by these investigations and indicated by the applied accuracy and reliability indicators we have to prefer definitely the arrangement E (images 3-4-7-8), because only in this case the x'- and y'- observations are of sufficient reliability together with a satisfactory accuracy of the object points.
Moreover, these investigations may show the necessity of performing a-priori accuracy and reliability studies for a proposed project. This becomes the more necessary the more non-transparent the geometrical conditions are. As a rule one should aspire reliability indicators which are at least equal or even greater than 0.6 (RI(x') $\geq$ 0.6, RI(y') $\geq$ 0.6) - in accordance with the values gained by the investigation of aerial triangulation systems (*Grün /11/*).
Naturally the global indicators don't provide in each practical case for sufficient accuracy / reliability of all individual object points / observations, e.g. if some observations have to be cancelled or if points cannot be observed from certain camera stations, whatever the reason may be for that. So the global investigation has often to be replaced by an individual checking (see (21)). To get further insight in detail problems and to become familiar with the data-snooping technique the arrangements of Figure 2 and Table 2 will be subject to the data-snooping procedure. To keep as close as possible to practice the synthetic image coordinate observations of all versions are superimposed by a random generator with means $\mu_{x'} = \mu_{y'} = 0$ and standard deviations $\sigma_{x'} = \sigma_{y'} = 5$ µm.
Then in all versions A - E different gross errors are introduced as (the gross

errors do exclusively appear at images no. 1 resp. 3 and at points no. 1 resp. 2):

Case a: One gross error at the image coordinate x' of control point no.1 of image 1 resp. 3: $\nabla x_1' = -30$ μm

Case b: Two gross errors simultaneously at the image coordinates x', y' of control point no.1 of image 1 resp. 3: $\nabla x_1' = \nabla y_1' = -30$ μm

Case c: One gross error at the image coordinate x' of non-control point no.2 of image 1 resp. 3: $\nabla x_2' = -30$ μm

Case d: One gross error at the image coordinate y' of non-control point no.2 of image 1 resp. 3: $\nabla y_2' = -30$ μm

With respect to case b we act as if only one gross error exists, although two gross errors have been introduced.
The following parameters are chosen resp. computed for testing:

$$\alpha_o = 0{,}001 \quad (0{,}1\%)$$

$$\beta(\lambda_p) = \beta(\lambda_1) = \beta_o = 0{,}8 \quad (80\%)$$

$$\lambda_o = 17.1 \quad , \sqrt{\lambda_o} = 4.14$$

Critical value for data-snooping (with $\alpha_o = 0{,}001$) from $t(1-\alpha_o, \infty)$: $c(w_i) = 3.29$.
Hence the minimal gross error which can be detected with the probability β_o is (with $P_B = I$):

$$\nabla l_1^{(i)} = \sigma_o \sqrt{\frac{\lambda_o}{q_{v_i v_i}}} = \frac{20{,}7}{\sqrt{q_{v_i v_i}}} \quad . \qquad (27)$$

Thus we obtain for the different arrangements and gross error situations the values of Table 3 for the minimal gross errors.

Table 3 : Minimal gross errors to be detected with the probability $\beta_o = 0{,}8$.

Arrangements	Case a $\nabla x_1'$(min) [μm]	Case b $\nabla x_1'$(min) [μm]	Case b $\nabla y_1'$(min) [μm]	Case c $\nabla x_2'$(min) [μm]	Case d $\nabla y_2'$(min) [μm]
A	28.3	28.3	27.2	-	32.0
B	31.4	31.4	29.6	-	32.4
C	29.6	29.6	27.7	48.5	27.3
D	25.5	25.5	25.2	32.4	25.9
E	27.2	27.2	26.8	31.3	26.2

From Table 3 we see that the minimal gross errors are grouped around 30 μm, except the cases Ac, Bc, Cc. That's the reason why we have introduced for all cases a-d a gross error of -30 μm.
After the computation of all error cases with all network arrangements we have got the results of Table 4 for the global test (H_o: $E(\hat{\sigma}_o^2) = \sigma_o^2$). Because the significance level α_o for data-snooping was kept constantly, the significance level α for

the global test - taken from *Baarda's* nomograms in */3/* - is varying from 0.22 until 0.45.

Table 4: Global test of the null-hypothesis H_o: $E(\hat{\sigma}_o^2) = \sigma_o^2$ for all network arrangements and gross error cases.

Arrangement	θ for gross error cases a	b	c	d	c(θ)	α	θ without gross err.
A	2.06	2.51	1.40	1.75	1.18	0.22	1.40
B	0.92	1.07	0.57	0.99	1.18	0.22	0.57
C	1.30	1.38	1.13	1.23	1.04	0.37	0.96
D	1.42	1.58	1.28	1.34	1.02	0.45	1.21
E	0.85	0.98	0.83	0.94	1.02	0.45	0.72

$$\theta = \frac{\hat{\sigma}_o^2}{\sigma_o^2} \quad , \quad c(\theta) \text{ from } F(1-\alpha, r, \infty)$$

Regarding the results of Table 4 reasonable doubt is put forward with respect to the practical application of the global test. In the gross error cases both versions B and E don't lead to the correct statement, i.e. to the rejection of the null-hypothesis. Though there are fairly good presumptions, because the expectation $E(\hat{\sigma}_o^2) = \sigma_c^2$ is a-priori known by the utilization of a random generator even under the null-hypothesis (when no gross errors are included) the critical values are exceeded twice. Of course the random generator produced discrete random values, whose standard deviation differs more ore less from the a-priori chosen value of 5 μm, but this problem is even intensified in real practical projects: What is then the expectation of $\hat{\sigma}_c^2$? That is probably the reason why *Pope /17/* doesn't use the global test - generally its power of assertion is not sufficient. Hence it is not recommended in practice to make the setup of individual tests (data-snooping) dependent on the result of the global test. The procedure of data-snooping should be applied in any case if reasonable suspicion exists that gross errors may appear, what is the standard situation in practical projects.
In this way we continue with the individual checking of the residuals. To avoid at least partly the disturbing effects of correlations between the residuals the strategy consists in rejecting those observations, for which the test criterion $w_i = \frac{|v_i|}{\sigma_{v_i}}$ is maximal.
As a result it was found out that all a-priori known gross errors have been detected, expect in the following cases:

1. Gross error case c in arrangement A and B: zero variance situation
2. Failure with the gross error case a of arrangement B:
   the x'-observation of point no. 3 at image 3 led to max (w_i) and thus to the prior rejection, although not including a gross error
3. Failure with the gross error case b of arrangement B:
   the x'-observation of point no. 9 at image 3 led to max (w_i), although not including a gross error

4. Failure with the gross error case c of arrangement D, where the test criterion $w_i = 3.15$ of the x'-observation of point no. 2 at image 1 remained slightly under the critical value of $c(w_i) = 3.29$

Analysing these failures it becomes evident that in 1. there is a-priori no chance for detection at all (zero variance problem). In 4. the situation was met, where the actual gross error (30µm) is smaller than the minimal gross error which can be detected with the probability $\beta_o = 0{,}8$ (see Table 4). Setting $\nabla x_2' = -32.4\mu m$ in version D the corresponding test criterion exceeds the critical value, thus leading to a correct rejection.
The false rejection of x_3' at image 3 in case Ba and of x_9' at image 3 in case Bb is probably a result of shiftings caused by correlations. Actually the presence of correlations disturbs significantly the statistical foundation of the gross error problem, so that correlations should be subject to future studies. The correlations become particularly of interest in a double sense:

- For the rejection procedure of data-snooping. The answer to the question whether only max (w_i) or all w_i which exceed the critical value should be rejected depends on the size of correlations.
- For the localization of gross errors the correlations are of decisive importance. Consider our practical examples: In the cases Ad, Bd the localization of $\nabla y_2'$ is impossible since the residuals $v_{y_2'}$ (image 1) and $v_{y_2'}$ (image 2) resp. $v_{y_2'}$ (image 3) and $v_{y_2'}$ (image 4) are perfectly correlating. Strong correlations exist also in case Cc between the x'-residuals of a non-control point in all three images. Here we obtain for the correlation coefficients of the x'-residuals of point no.2:

  $$r_{v_{x'}[3]\,v_{x'}[0]} = -0.960 \ , \quad r_{v_{x'}[3]\,v_{x'}[4]} = 0.956 \ , \quad r_{v_{x'}[0]\,v_{x'}[4]} = -0.960$$

  (image number in brackets)

  with the result that the gross error $\nabla x_2'$ cannot be localized.

Experiences gained so far with the data-snooping technique are very encouraging, expecially if its results are compared with conventional, non-statistical methods of gross error detection.
From all 23 gross error cases of our examples (the 2 zero variance cases are excluded) we obtained by conventional methods and by data-snooping:

$|v| > 3\sigma_o$: 15 correct rejections, 12 incorrect rejetions

$|v| > 3\hat{\sigma}_o$: 10 correct rejections, 5 incorrect rejections

data-snooping: 19 correct rejections, 2 incorrect rejections

So it becomes an important and promising task to do further investigations in this field of research to arrive lately at systems for point determination providing for both accuracy and reliability.

D. Statistical test criterions for parameter testing in photogrammetric point determination problems

Recently photogrammetry uses more and more the methods of statistical interval estimation and hypotheses testing. Thereby statistical test criterions are sometimes applied in a little careless manner, what may often lead to wrong decisions concerning the acceptance or rejection of a hypothesis.
So the problem of statistical hypotheses testing is treated in a more general way and some test criterions are derived which are often useful for the analysis of results obtained by methods of photogrammetric point determination.
At any rate one should always keep in mind that a statistical test cannot be an end in itself, it cannot serve as an inevitable standard, but must only be regarded as an aid for decision.
For hypotheses testing we suppose the linear model of self-calibrating bundle adjustment, derived from (1)

$$E(l) = Ax \quad , \qquad E(e) = 0 \quad , \quad D(e) = D(l) = \sigma_o^2 P^{-1} \quad , \tag{28}$$

with the general linear null-hypothesis of full rank

$$H_o: \quad Bx = w \quad , \tag{29}$$

where B is a $b \times u$- matrix with $\mathrm{rank}(B) = b \leq u$.
The minimum variance unbiased estimators for x and σ_o^2 from (28) are with $r = n-u$

$$\hat{x} = (A^T PA)^{-1} A^T Pl \tag{30a}$$

$$\hat{\sigma}_o^2 = \frac{1}{r} (A\hat{x}-l)^T P(A\hat{x}-l) \quad . \tag{30b}$$

For statistical analysis the distribution of l is assumed as $l \sim N(Ax, \sigma_o^2 P^{-1})$, i.e. a n-dimensional normal distribution.
Testing H_o requires the derivation of a test function T such that the distribution of T is known when $Bx = w$ (i.e. the null-hypothesis is true).
Functions with optimal properties are the Likelihood ratio functions. Their application leads to the test criterion

$$T = \frac{R}{b\,\hat{\sigma}_o^2} \quad , \tag{31}$$

with

$$R = (B\hat{x}-w)^T (B(A^T PA)^{-1} B^T)^{-1} (B\hat{x}-w) \quad . \tag{32}$$

For the derivation of T see *Graybill /9/, Koch /13/.*
If H_o is true, then T is distributed as Snedecor's $F(b,r)$.
To obtain the power of the test H_o against an alternative hypothesis H_A the distribution of T must also be known, if the alternative hypothesis H_A: $Bx \neq w$ ($B'x = w'$) is true. Then we find T distributed as the noncentral $F'(b,r,\lambda)$ with the noncentrality parameter

$$\lambda = \frac{1}{2\sigma_o^2} (B'x-w)^T (B(A^T PA)^{-1} B^T)^{-1} (B'x-w) \quad . \tag{33}$$

The power $\beta(\lambda)$ of the test may be computed from Tang's tables of the noncentral beta distribution (*Graybill /9/*).

The general linear hypothesis (29) respectively some of its simplifications can be applied to quite different problems of photogrammetric point determination. In this paper we will restrict on three test problems, often appearing in close-range photogrammetry, as significance testing of

- additional parameters for systematic error estimation
- residuals at check points and control points for empirical accuracy studies
- coordinate differences in deformation measurements (movement studies).

1. Significance testing of additional parameters

To check the statistical significance of an additional parameter set, i.e. to test whether a systematic influence does occur at all, we have to construct a global test for the whole set of additional parameters. For that purpose we put with respect to the nomenclature of the unknowns in (1):

$$H_o: \underset{p,p}{I} \; \underset{p,1}{dz} = \underset{p,1}{dz'} \quad . \qquad \begin{array}{l} I = \text{Identity matrix} \\ p = \text{number of additional parameters} \end{array} \tag{34}$$

That is, we compare simultaneously each individual additional parameter dz_k ($k = 1, \ldots, p$) with a supposed value dz'_k. If no further informations are available, e.g. from precalibrations, $dz' = 0$ will be chosen.
Thus the test criterion T becomes

$$T = \frac{1}{p\hat{\sigma}_o^2} (d\hat{z}-dz')^T Q_{zz}^{-1} (d\hat{z}-dz') \quad . \tag{35}$$

According to (1) the symmetric matrix N of the normal equations is

$$N = \left(\begin{array}{ccc|c} A_1^T P_B A_1 & A_1^T P_B A_2 & A_1^T P_B A_3 & A_1^T P_B A_4 \\ ./. & A_2^T P_B A_2 + P_p & A_2^T P_B A_3 & A_2^T P_B A_4 \\ ./. & ./. & A_3^T P_B A_3 & A_3^T P_B A_4 \\ \hline ./. & ./. & ./. & A_4^T P_B A_4 + P_Z \end{array}\right) = \begin{pmatrix} N_{xx} & N_{xz} \\ N_{xz}^T & N_{zz} \end{pmatrix} \tag{36}$$

Then

$$Q_{zz}^{-1} = N_{zz} - N_{xz}^T N_{xx}^{-1} N_{xz} \tag{37}$$

is the partly reduced matrix of the normal equations N for the set of additional parameters. If the additional parameters are arranged at the end of the complete vector of unknowns as it is supposed in (1), then Q_{zz}^{-1} is obtained without additional effort during the triangular factorization of N.
The noncentrality parameter for the test H_o is

$$\lambda = \frac{1}{2\sigma_o^2} (dz-dz')^T \; Q_{zz}^{-1} (dz-dz') \quad , \tag{38a}$$

or

$$\lambda = \frac{1}{2\sigma_o^2} (dz-dz')^T N_{zz} (dz-dz') - \frac{1}{2\sigma_o^2} (dz-dz')^T N_{xz}^T N_{xx}^{-1} N_{xz} \; (dz-dz') \quad . \tag{38b}$$

The power $\beta(\lambda)$ is an increasing function of λ, so λ should be as large as possible. λ is maximal if $N_{xz} = 0$ (*Graybill /9/*), i.e. if dz is orthogonal to dx, dx^P, dt Thus a statistical advantage of the orthogonal additional parameter concept becomes evident: It increases the power of this test.
Beside this it is clear that the power of a test depends mainly on the alternative hypothesis and on the type I error size α.
If H_o is rejected, i.e. if a significant systematic influence is apparent, the detection of individual significant components becomes necessary.
Therefore the set of hypotheses

$$H_o^{(i)} : dz_i = dz_i' \quad , \qquad (i = 1, \ldots, p) \tag{39}$$

is commonly used to test the individual parameters on significance.
Hence we get the test criterions

$$T^{(i)} = \frac{(d\hat{z}_i - dz_i')^2}{\hat{\sigma}_o^2 \, q_{z_i z_i}} \quad . \qquad q_{z_i z_i} = \text{variance of the ith additional parameter} \tag{40}$$

Under $H_o^{(i)}$ the $\sqrt{T^{(i)}}$- values are distributed as Student's t with r degrees of freedom. Thus the more-dimensional T-test (31) is reduced to an one-dimensional t-test.
In the case of independence of the individual parameters the type I error size $\bar{\alpha}$ of the individual test is related to the type I error size α of the global test as

$$1 - \alpha = (1 - \bar{\alpha})^p \quad , \tag{41}$$
$$\bar{\alpha} = \frac{\alpha}{p} \quad .$$

The main problem in testing subsets or even single parameters estimated in multi- ensional models arises with the dependence of these subsets (single parameters) the other model parameters. In the case of significant correlations the probability $\bar{\alpha} = P(T^{(i)} > c_o \mid H_o^{(i)})$ of rejection of a true null-hypothesis $H_o^{(i)}$: $dz_i = dz_i'$ of a single event (parameter) is no further independent. It should be noticed that the application of the one-dimensional t-test with the usual limits and confidence intervals leads the more to wrong decisions the more the correlations do increase. Thus we have another important argument for using the concept of orthogonal additional parameter sets.

Whereas the orthogonal concepts are regarded as very useful, the practical geometrical conditions however often do not provide for sufficient orthogonality. Then we have to set up simultaneous confidence intervals for the single events. This can be done by the a-posteriori orthogonalization of the additional parameter set, i.e. the transformation of the additional parameter vector (or, if necessary, even the vector of all unknowns of the bundle system (1)) into orthogonal components, which can be tested independently (see *Roy, Bose /13/, Pelzer /12/*).
Sometimes high correlations do appear only within a subset of the additional parameters. Then these parameters can be tested together on common significance. The corresponding test criterion may analogously be derived from the general li-

near hypothesis (29).
Here a bivariate test criterion is derived, which can be applied if two additional parameters are highly correlating.
The null-hypothesis is formulated as

$$H_o : \underset{2,2}{I} \underset{2,1}{dz} = \underset{2,1}{dz'} \tag{42}$$

Suppose $dz' = 0$ and $dz^T = (dz_i, dz_k)$, then T results in

$$T = \frac{1}{2\hat{\sigma}_o^2} (d\hat{z}_i \; d\hat{z}_k) \begin{pmatrix} q_{z_i z_i} & q_{z_i z_k} \\ q_{z_k z_i} & q_{z_k z_k} \end{pmatrix}^{-1} \begin{pmatrix} d\hat{z}_i \\ d\hat{z}_k \end{pmatrix} . \tag{43}$$

The application of various test criterions and a possible strategy for additional parameter rejection was demonstrated in *Grün* /20/ with the use of a practical aerial triangulation example. Because of the extension of aerial triangulation systems one has to work with some neglectings concerning the application of completely rigid statistical tests; thus the concept of a-posteriori orthogonalization of the whole system is hardly to realize. However, close-range systems open a wide and interesting field of application of sophisticated statistical tests. So this subejct should be a main objective for further studies.

2. Significance testing of residuals at check points and control points

It is a widespread and useful procedure in empirical accuracy studies to compare the set of coordinates of a photogrammetric adjustment with their "true" values, mostly obtained by more precise observation methods. Then the r.m.s. values of the residuals serve as absolute accuracy measures to check the photogrammetrically achieved accuracy.
If we denote the estimated residual vectors at check points as

$$\begin{aligned} \Delta\hat{X}^T &= (\Delta\hat{X}_1, \ldots, \Delta\hat{X}_{p_L}) , \\ \Delta\hat{Y}^T &= (\Delta\hat{Y}_1, \ldots, \Delta\hat{Y}_{p_L}) , \\ \Delta\hat{Z}^T &= (\Delta\hat{Z}_1, \ldots, \Delta\hat{Z}_{p_Z}) , \end{aligned} \qquad \begin{aligned} p_L &= \text{number of planimetric check points} \\ p_Z &= \text{number of height check points} \end{aligned} \tag{44}$$

with the components

$$\begin{aligned} \Delta\hat{X}_i &= X_i^s - \hat{X}_i^{Ph} , \\ \Delta\hat{Y}_i &= Y_i^s - \hat{Y}_i^{Ph} , \\ \Delta\hat{Z}_k &= Z_k^s - \hat{Z}_k^{Ph} , \end{aligned} \qquad \begin{aligned} & i = 1, \ldots, p_L \\ & k = 1, \ldots, p_Z \\ & \hat{X}^{Ph}, \hat{Y}^{Ph}, \hat{Z}^{Ph} = \text{estimated photogrammetric coordinates} \\ & X^s, \; Y^s, \; Z^s = \text{"true" coordinates} \end{aligned} \tag{45}$$

then the r.m.s. values are defined as

$$\begin{aligned} & \hat{\mu}_X^2 = \frac{\Delta\hat{X}^T\Delta\hat{X}}{p_L} , \quad \hat{\mu}_Y^2 = \frac{\Delta\hat{Y}^T\Delta\hat{Y}}{p_L} , \quad \hat{\mu}_Z^2 = \frac{\Delta\hat{Z}^T\Delta\hat{Z}}{p_Z} , \\ & \hat{\mu}_{X,Y}^2 = \frac{\hat{\mu}_X^2 + \hat{\mu}_Y^2}{2} . \end{aligned} \tag{46}$$

Molenaar /15/ has proved that in the case of only random influences the estimators $p_L\hat{\mu}_X^2$, $p_L\hat{\mu}_Y^2$, $p_Z\hat{\mu}_Z^2$, $2p_L\hat{\mu}_{X,Y}^2$ are indeed unbiased, but not sufficient estimators for the corresponding parts of the trace of the variance-covariance matrix of the estimated check point coordinates.
The r.m.s. errors (estimators) are often accepted to be unbiased without any statistical investigation or from the plots of the residuals a systematic deformation is stated or not. *Molenaar /15/* has shown, that a sophisticated statistical treatment is possible to get a better decision basis. He interprets the photogrammetric adjustment as an adjustment in steps and is thus deriving the test criterions.
In the following the corresponding test criterions are derived with help of the concept of the general linear hypothesis (29), which seems to be a little bit clearer and which makes evident at the same time a suitable computational strategy.
We suppose the statistical model (28) and start from the null-hypothesis that the common residual vector $d\hat{X}^T = (\Delta\hat{X}^T, \Delta\hat{Y}^T, \Delta\hat{Z}^T)$ is multi-dimensional normally distributed with $d\hat{X} \sim N(dX, \sigma_o^2 Q_{XX})$. Q_{XX} is the weight coefficient matrix of the estimated check point coordinates.
Thus H_o becomes

$$H_o: \underset{p,p}{I} \underset{p,1}{dX} = \underset{p,1}{X^S} - \underset{p,1}{X^N} \quad , \qquad p = 2p_L + p_Z = \text{total number of check point coordinates} \tag{47}$$

with

dX = sub-quantity for the check points of dx (see (1))
X^S = "true" check point coordinates vector $X^{S^T} = (X^{S^T}, Y^{S^T}, Z^{S^T})$
X^N = vector of the approximate values for the check point coordinates in (1)

Mostly $X^S - X^N = 0$ can be introduced.
Hence we get the test criterion

$$T = \frac{R}{p \cdot \hat{\sigma}_o^2} \quad , \tag{48}$$

with

$$\begin{aligned} R &= (d\hat{X}-(X^S-X^N))^T Q_{XX}^{-1} (d\hat{X}-(X^S-X^N)) = \\ &= (\hat{X}^{Ph}-X^S)^T Q_{XX}^{-1} (\hat{X}^{Ph}-X^S)) \quad . \end{aligned} \tag{49}$$

If in model (1) the check point coordinates are arranged at the end of the complete solution vector, then Q_{XX}^{-1} is obtained without additional effort during the triangular factorization of $(A^T PA)$ - see Section D1. Under H_o T is distributed as the central F (p,r). The power of the test depends on the alternative hypothesis and may be computed as mentioned before. Mostly it is advisable to base the alternative hypothesis on the assumption of a polynomial deformation. In any case the rejection of the null-hypothesis should give rise to examine both the additional parameter model (or even to introduce such a model) and the control and check point accuracy.
It is easy to see how the test criterions for special coordinate vectors, e.g. X, Y independent on Z, can be derived.

To obtain a better insight in the character of a systematic deformation it is necessary to test individual groups of residuals or even single residuals. For this purpose appropriate test criterions can also be derived as it was demonstrated for additional parameters in section D1. But notice the difficulties arising with high correlations with respect to the confidence interval setup.
It is an unpleasant experience that the inadequate presicion of control points does often disturb the photogrammetrically obtained coordinates, thus leading to wrong conclusions with respect to the photogrammetric accuracy potential. The disturbing influence of control points may be caused randomly or systematically. However, a statistical treatment of the residuals at control points is a helpful device to arrive at better conclusions with respect to the control accuracy in any case.
Of course it is not easy or often impossible to seperate the photogrammetric effect from the control influence and to seperate the random control errors from the systematic control errors. But if a self-calibrating adjustment is performed we can assume that the main photogrammetric systematic influence is compensated, so that the residuals at control points may indicate at least a part of the original systematic control deformation.

A global test on the significance of control residuals can be based on

$$H_0: \underset{d,d}{I} \; \underset{d,1}{dx^P} = \underset{d,1}{dx^{P'}} \quad , \qquad d = \text{number of control coordinates} \tag{50}$$

(Mostly $dx^{P'}$ will be assumed as $dx^{P'} = 0$).

Hence we get the test criterion

$$T = \frac{R}{d \cdot \hat{\sigma}_o^2} \quad , \tag{51}$$

with

$$R = (d\hat{x}^P - dx^{P'})^T Q_{pp}^{-1} (d\hat{x}^P - dx^{P'}) \quad . \tag{52}$$

Under H_o T is distributed as the central $F(d,r)$.

Usually the weight matrix P_p of the control point coordinates (see(1)) will be a diagonal matrix. If the weights are not too small then $Q_{pp}^{-1} - Q_{pp}$ is the weight coefficient matrix of the estimated control coordinates $\bar{p}$ is diagonal dominant, i.e. the coefficient columns of the control residuals dx_i are mostly orthogonal to the other unknowns of system (1).
So the application of the t-test for the testing of individual residuals dx_i^P with a type I error size $\bar{\alpha}$:

$$1 - \alpha = (1 - \bar{\alpha})^d \quad , \qquad \alpha = \text{type I error size of the global test (50)} \tag{53}$$

is more successfull than in the cases mentioned before.

3. Significance testing of coordinate differences in deformation measurements

The determination of point movements requires repetitive object measurements to various periods. If the observation program is exactly identical during the different periods the multivariate concept is an appropriate model for point estimation, interval estimation and hypotheses testing.
Generally the corresponding repetitive observations are not independent. In this case the covariance matrix of these observations can be estimated immediatly - on the contrary to the univariate adjustment, where usually only the variance of unit weight is estimated.
For multivariate estimation problems and hypothesis testing see *Anderson /1/, Koch /14/*. Statistically seen, the multivariate concept is the most general one, but it requires design matrices which are identically for each observation period. Since this is an essential practical restriction for photogrammetric problems - observations can only be rejected if this is done for those belonging together in all periods, the camera stations and the rotation elements are not allowed to be changed - we use an univariate model, adapted to the special situation of deformation measurements.
Without loss of generality we restrict on two observation periods I, II. Thus we get the model

$$
\begin{aligned}
& -e = A_I x_I + A_{II} x_{II} - l \quad ; \quad P \\
& e = \begin{pmatrix} e_I \\ e_{II} \end{pmatrix}, \quad l = \begin{pmatrix} l_I \\ l_{II} \end{pmatrix}, \quad P = \begin{pmatrix} P_I & 0 \\ 0 & P_{II} \end{pmatrix} \\
& E(l) = A_I\, x_I + A_{II}\, x_{II} \quad , \\
& D(e) = D(l) = \sigma_o^2\, P^{-1} \quad ; \qquad E(e) = 0 \quad .
\end{aligned}
\tag{54}
$$

Model (54), which is an extension of model (28), permits various arrangements of the unknowns. For example, the solution vector of the control points can be regarded as a joint vector for both periods. The same may yield for the vector of additional parameters. If no common unknowns are used, system (54) can be divided into two seperate adjustments.
According to the purpose of movement determination we set up the globel null - hypothesis (with model (54) and the notation of (1)) to

$$
H_o : \underset{3m,6m}{B}\ \underset{6m,1}{dx} = \underset{6m,1}{w} \quad , \qquad m = \text{number of points to be jointly tested} \tag{55}
$$

The structure of B depends on the sequence of the unknowns in model (54). If the vector of point coordinates to be tested is arranged like

$$
dx^T = (dX_{I1}, dX_{II1}, dY_{I1}, dY_{II1}, dZ_{I1}, dZ_{II1}, \ldots, dX_{Im}, dX_{IIm}, dY_{Im}, dY_{IIm}, dZ_{Im}, dZ_{IIm})
$$

then B is structured as

$$
\underset{3m,6m}{B} = \begin{pmatrix} 1 & -1 & 0 & \cdots & & 0 \\ 0 & 1 & -1 & 0 & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & 0 & 1 & -1 \end{pmatrix} \quad . \tag{56}
$$

dx = vector of coordinates belonging together in period I and II to be jointly tested.

The test criterion $\bar{T}$ results in

$$T = \frac{R}{3m\ \hat{\sigma}_o^2} \quad , \tag{57}$$

with

$$R = (d\hat{x}_I - d\hat{x}_{II} - w)^T \ (B\ Q_{xx} B^T)^{-1} \ (d\hat{x}_I - d\hat{x}_{II} - w) \quad . \tag{58}$$

$$dx_I^T = (dX_{I1},\ dY_{I1},\ dZ_{I1},\ \ldots,\ dX_{Im},\ dY_{Im},\ dZ_{Im}) \quad ,$$

$$dX_{II}^T = (dX_{II1},\ dY_{II1},\ dZ_{II1},\ \ldots,\ dX_{IIm},\ dY_{IIm},\ dZ_{IIm}) \quad ,$$

Q_{xx} = joint weight coefficient matrix of the point coordinates to be tested

Mostly $w = 0$ has to be chosen (especially if equal approximate values are used)

Again each admissible sub-hypothesis can be tested. Here it becomes particularly important to test for significant deformations at individual points or even at special coordinates of individual points. After all what has been shown before it should be an easy exercise to derive those special null-hypotheses and the corresponding test criterions.

On account of the purpose of deformation studies it becomes very important in this case to apply correct confidence intervals. Hence the a-posteriori orthogonalization is an indispensable demand to eliminate the disturbing effects of high correlations.

E. Concluding remarks

The relatively small extension of the linear equations of close-range systems often enables the application of sophisticated linear models, estimation procedures and statistical techniques. This fact should be utilized more than it was done by now to obtain best possible results with respect to accuracy and reliability. Like in aerial triangulation systems the self-calibrating bundle method can provide for significant model improvements and thus for remarkable accuracy enhancements, if suitable and highly developed additional parameter sets are applied, as presented in this paper with the concept of orthogonal bivariate polynomials. Practical experiences with aerial triangulation systems show however, that the functional extension of a bundle model by additional parameters must be attended by comprehensive statistical test procedures to protect against overparametrization. Actually statistical methods should find more attention in close-range applications, especially for test field analysis and in deformation measurements.

A really virgin area but nevertheless a very important subject in close-range

photogrammetry is the statistical treatment of reliability problems.
As it was shown in this paper a statistically founded strategy for gross error detection provides for a remarkable efficiency.
Suitable approaches for gross error detection on a statistical base are available and should obtain more attention in the near future.
Thus photogrammetric close-range systems will reach an accuracy standard and a reliability level which provide for outstanding results in each practical project.

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